Solving Linear Systems of Equations Over Cyclotomic Fields

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We want to solve large linear systems involving roots of unity arising from a problem in computational group theory. For example, the complex number i satisfies $i^4 = 1$. It is a primitive 4^{th} root of unity. A primitive k^{th} root of unity is a root of the cyclotomic polynomial $m_k(z)$. For example, i is a root of $m_4(z) = z^2 + 1$. The cyclotomic polynomials are of special interest because there are lots of primes for which $m_k(z)$ factors into distinct linear factors modulo p. For example

$$m_5(z) = z^4 + z^3 + z^2 + z + 1 = (z-3)(z-4)(z-5)(z-9) \bmod 11.$$

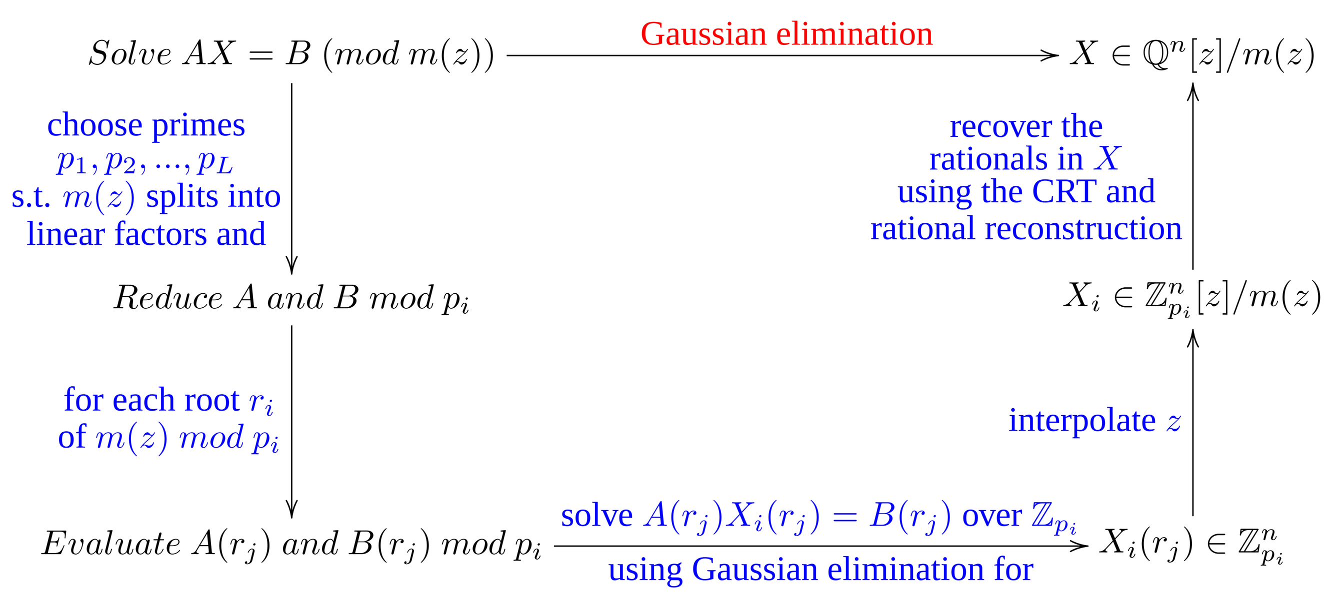
The following lemma tells us how to find such primes and how to factor $m_k(z)$.

Lemma: If p is a prime and $k|(p-1)$, then $m_k(z)$ has $d = \deg\ m(z)$ roots in $\mathbb{Z}_p$. Moreover, if ω is a primitive k^{th} root of unity $mod\ p$, then

$$\{\omega^i : 1 \le i \le k \text{ and } gcd(i,k) = 1\}$$

are roots of $m_k(z)$.

We exploit this to design two efficient algorithms for solving a linear system $AX = B$ involving roots of unity. The following figure describes the first algorithm:



Algorithm 1: Chinese Remaindering with Rational Reconstruction

Input: Matrix $A \in \mathbb{Q}^{n\times n}[z]$, vector $B \in \mathbb{Q}^n[z]$, polynomial $m(z) \in \mathbb{Z}[z]$.
Output: Vector $X \in \mathbb{Q}^n[z]$ which satisfies $AX = B\ (mod\ m(z))$.

1. Clear fractions in A and B and set $k := 1$.
2. **Loop**
   (a) Pick a new prime p_k which splits $m(z)$.
   (b) Find all roots $r_1, \ldots, r_d$ of $m(z)$ mod p_k.
   (c) **for** $j = 1$ **to** d **do**
      Solve $A(r_j)X_k(r_j) \equiv B(r_j)$ mod p_k for $X_k(r_j) \in \mathbb{Z}^n_{p_k}$.
      If there is no solution then go back to step $2(a)$.
   (d) Interpolate $X_k(z) \in \mathbb{Z}^n_{p_k}[z]$ from points r_j's and $X_k(r_j)$'s.
   (e) Apply Chinese remaindering to recover X mod $p_1 \times p_2 \times \cdots \times p_k$.
   (f) Apply rational reconstruction to recover $X \in \mathbb{Q}^n[z]/m(z)$.
   (g) If $m(z)|AX - B$ then output X.
   (h) Set $k := k + 1$.

Theorem 1: The running time of above algorithm is $O(n^2dLc + n^2d^2L + n^3dL + ndL^2)$ where $L \in O(ndc)$ is the number of primes needed, $n = \dim\ A$, $d = \deg\ m(z)$, $c = \max(\log ||A||_\infty, \log ||B||_\infty)$.

The above running time does not include the trial division in step $2(g)$ since this step may be avoided if we use sufficiently many primes. Solving a linear system $mod\ p_k$ using Gaussian elimination brings a factor of n^3, and rational reconstruction and Chinese remaindering brings a factor of L^2 which are the two main costs of the algorithm. An improvement can be made by doing rational reconstruction and trial division only after $1, 2, 4, 8, 16, \ldots$ primes.

Algorithm 2: p-adic Lifting with Rational Reconstruction

Input: Matrix $A \in \mathbb{Q}^{n\times n}[z]$, vector $B \in \mathbb{Q}^n[z]$, polynomial $m(z) \in \mathbb{Z}[z]$.
Output: Vector $X \in \mathbb{Q}^n[z]$ which satisfies $AX = B\ (mod\ m(z))$.

1. Clear fractions in A and B.
2. Pick a prime p which splits $m(z)$ and find all roots $r_1, \ldots, r_d$ of $m(z)$ mod p.
3. Set $k := 1$, $error := B$ and compute $A^{-1}(r_i) \bmod p$ for $i = 1, 2, \ldots, d$.
   If $A(r_j)$ is not invertible (mod p) then go back to step 2 and pick a new prime.
4. **Loop**
   (a) **for** $j = 1$ **to** d **set** $X_{k-1}(r_j) = A^{-1}(r_j)error(r_j) \bmod p$.
   (b) Interpolate $X_{k-1}(z) \in \mathbb{Z}^n_p[z]$ from r_i's and $X_{k-1}(r_j)$'s.
   (c) Compute $error := (error - AX_{k-1})/p$.
   (d) Obtain $X^{(k)} := X^{(k-1)} + X_{k-1}p^{k-1} = X_0 + X_1 \times p + X_2 \times p^2 + \cdots + X_{k-1} \times p^{k-1}$.
   (e) Apply rational reconstruction to recover $X \in \mathbb{Q}^n[z]/m(z)$.
   (f) If $m(z)|AX - B$ then output X.

Theorem 2: The running time of above algorithm is $O(n^3d + n^2d^2c^2 + nd^2cL + n^2dcL + ndc^2L + ndL^2)$ where $L \in O(ndc)$ is the number of lifting iterations needed, $n = \dim\ A$, $c = \max(\log ||A||_\infty, \log ||B||_\infty, \log n, d \log(||m||_\infty + 1))$, $d = \deg\ m(z)$.

The same remarks made about trial division apply here. The most costly part of this algorithm is updating the error in step $4(c)$. We implemented two variations which reduce the cost. (See **Lift 1** and **Lift 2** below) Also, this algorithm does d Gauss eliminations in step 3 whereas the first does Ld in step $2(c)$ which is why it is faster for large n.

Timings (in CPU seconds) for random systems.

n	Coefficient Length c							Remark
	2 digits	4 digits	8 digits	16 digits	32 digits	64 digits	128 digits	
5	.303	.321	.340	.390	.472	.700	1.558	GE
	.019	.029	.069	.136	.312	.643	1.412	CRT
	.028	.027	.049	.102	.245	.631	1.797	Lift 1
10	1.947	2.185	2.375	2.744	3.623	6.210	15.317	GE
	.050	.097	.183	.418	1.019	2.359	5.685	CRT
	.058	.091	.152	.309	.803	2.084	6.384	Lift 1
20	16.041	17.927	20.759	26.141	37.817	71.288	186	GE
	.167	.347	.727	1.616	4.759	12.149	30.983	CRT
	.158	.276	.521	1.054	3.005	8.219	26.581	Lift 1
40	148	181	207	291	476	1033	2829	GE
	.797	1.795	3.899	8.756	31.120	85.780	234	CRT
	.500	.973	1.932	3.998	11.891	33.412	113	Lift 1
$m(z) := z^6 + z^5 + z^4 + z^3 + z^2 + z + 1,\ d = 6$								

Timings (in CPU seconds) for Dabbaghian's systems.

file	sys49	sys100	sys100b	sys144	sys196	sys225	sys256	sys576	sys900	sys900b
$deg_z(m)$	4	8	4	2	2	4	4	6	8	2
k	5	24	8	4	3	5	12	7	24	4
$\|\|A\|\|_\infty$	10	5	2	4	11	2	3	3	2	5
$\|\|x\|\|_\infty$	45	14	1	1	229	875	2	1	2	1
CRT	.144	.788	.029	.036	3.344	3.056	.155	.842	2.358	1.458
Lift 1	.109	.443	.030	.029	1.183	2.374	.174	.612	2.761	.462
Lift 2	.111	.294	.100	.163	1.973	1.678	.640	3.022	7.627	5.711
GE	109	3080	30.15	10.49	4419	769	848	2055	2265	1195
# primes	4	1	1	1	9	36	1	1	1	1
Det	.293	4.159	.305	.147	6.206	4.644	3.748	53.69	338	25.74

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