

Sparse Polynomials

November 23, 2023 7:06 PM

Let $f \in R[x_1, \dots, x_n]$, R a ring.

Let $f = \sum_{i=1}^t a_i \cdot M_i(x_1, \dots, x_n)$ where $a_i \in R$, $a_i \neq 0$.
 \uparrow
 monomials.

If $\deg(f, x_i) = d_i$ then f may have upto $t = \prod_{i=1}^n (d_i + 1)$ terms.

E.g. $n=2$, $d_1=3$, $d_2=2$. $t \leq (3+1)(2+1) = 12$.

		$\deg x_2$
		0 1 2
$\deg x_1$	0	1 x_2 x_2^2
1	x_1	$x_1 x_2$
2	x_1^2	$x_1^2 x_2$
3	x_1^3	$x_1^3 x_2$

We say f is sparse if

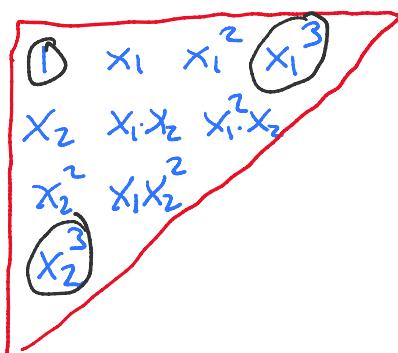
$$t = \#f \ll \prod_{i=1}^n (d_i + 1).$$

$$\therefore t \leq \sqrt{\prod_{i=1}^n (d_i + 1)}.$$

$$\text{Eg. } f = 3x_1 + 2x_1 x_2 + 5x_2 x_1^2$$

Alternatively if $\deg(f) = d$ then f may have upto $\binom{n+d}{d}$ terms.

E.g. $d=3, n=2$ $\binom{n+d}{d} = \binom{2+3}{3} = \frac{5!}{2!3!} = \frac{5 \cdot 4}{2} = 10$ terms.



We say f is sparse if
 $t \ll \binom{n+d}{d}$.

$$\text{I use } t \leq \sqrt{\binom{n+d}{d}} = \sqrt{10}$$

$$\text{So } f = 1 - x_1^3 - x_2^3 \text{ is sparse.}$$

$$T_3 = \begin{bmatrix} u & v & w \\ v & u & v \\ w & v & u \end{bmatrix} \quad \det(T_3) = u^3 - 2uv^2 - uw^2 + 2vw^2. \quad t = 4.$$

$$\binom{n+d}{d} = \binom{6}{3} = 20. \quad \uparrow \text{sparse.}$$

In still most polynomials with $n \geq 3$ are sparse.

Sparse Polynomial Interpolation.

Let $f \in k[x_1, \dots, x_n]$ where k is a field.

Suppose f is given by a black-box $B: k^n \rightarrow k$.

Suppose $f = \sum_{i=1}^t a_i M_i(x_1, \dots, x_n)$.

How many points in k^n do we need to interpolate f ?

I.e. to recover the $a_i \in k$ and the monomials M_i .

Let $d_i = \deg(f, x_i)$.

Since $\#f \leq \prod_{i=1}^n (d_i + 1) = D$ we may need D values of f .

$$\text{E.g. } f(x_1, x_2) = \sum_{j=0}^{d_1} \left(\sum_{i=0}^{d_2} a_{ij} x_2^i \right) \cdot x_1^j$$

Pick $\beta_0, \beta_1, \dots, \beta_{d_2} \in k$.

$$\begin{aligned} \text{Interpolate } f(x_1, \beta_0) &= \bullet + \bullet x_1 + \dots + \bullet x_1^{d_1} && \text{using } d_1+1 \text{ points for } x_1 \\ f(x_1, \beta_1) &= \bullet + \bullet x_1 + \dots + \bullet x_1^{d_1} && \text{using } d_1+1 \text{ points for } x_1 \\ &\vdots && \\ f(x_1, \beta_{d_2}) &= \bullet + \bullet x_1 + \dots + \bullet x_1^{d_1} && \text{using } d_1+1 \text{ points.} \end{aligned}$$

$$\Rightarrow f(x_1, x_2) = f_0(x_2) + f_1(x_2) \cdot x_1 + \dots + f_{d_2}(x_2) \cdot x_1^{d_1} \quad \text{So } (d_2+1)(d_1+1) \text{ points}$$

Suppose f is sparse. Can we improve on this?

$$\#\text{values of } f \quad f = 1 \cdot x_1 + 1 \cdot x_2 + \dots + 1 \cdot x_n$$

$$\text{Dense method.} \quad \prod_{i=1}^n (d_i + 1) \quad = (1+d)^n \leftarrow \text{is exponential in } n.$$

$$\text{Zipper}^{\text{PhD}} 1979 \quad \leq \left(t \sum_{i=1}^n d_i \right) + 1 \quad = t \cdot nd + 1$$

$$\text{Ben-Or/Tiwari}^{\text{PhD}} 1988 \quad 2T \quad \text{where } T \geq ? \quad 2T$$