POLY : A new polynomial data structure for Maple 17 that improves parallel speedup.

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Parallel Computer Algebra Applications
ACA 2012, Sofia, Bulgaria
June 25-28, 2012

This is joint work with Roman Pearce.

Talk Outline

- Polynomial data structures in Maple and Singular are slow.
  Our data structure.

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- Why is parallel speedup poor?
  Solution and new timings.

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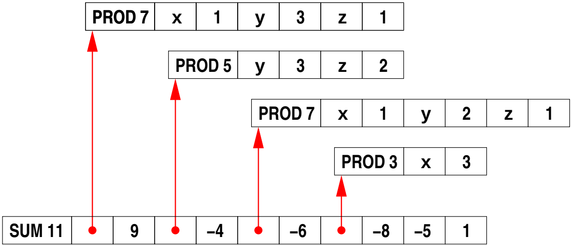
- Why is parallel speedup poor?
  Solution and new timings.
- Notes on integration into Maple kernel for Maple ≥ 17.
- Conclusion

Representations for $9\,xy^3z - 4\,y^3z^2 - 6\,xy^2z - 8\,x^3 - 5$.

Maple 16



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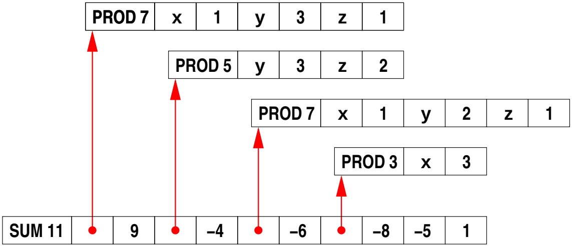
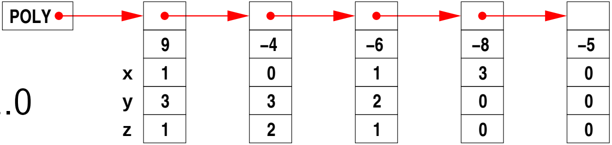


Singular 3.1.0



- Memory access is not sequential.
- Monomial multiplication costs O(100) cycles.

Our representation $9\,xy^3z - 4\,y^3z^2 - 6\,xy^2z - 8\,x^3 - 5$.

SEQ 4	x	y	z

POLY 12	•	5131	9	5032	−4	4121	−6	3300	−8	0000	−5

Monomial encoding for graded lex order with $x > y > z$

Encodes $x^i y^j z^k$ in a single word $\boxed{d \mid i \mid j \mid k}$ where $d = i+j+k$.

Advantages

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Monomial encoding for graded lex order with $x>y>z$

Encodes $x^iy^jz^k$ in a single word $\boxed{d \mid i \mid j \mid k}$ where $d = i+j+k$.

Advantages

- It's more compact.
- Memory access is sequential.
- Fewer objects to clutter tables.
- Monomial $>$ and $\times$ cost **one** instruction.

Multiplication using a binary heap.

Let $f = f_1 + f_2 + \cdots + f_{\mathbf{n}}$ and $g = g_1 + g_2 + \cdots + g_{\mathbf{m}}$.
Compute $f \times g = f_1 \cdot g + f_2 \cdot g + \cdots + f_n \cdot g$.

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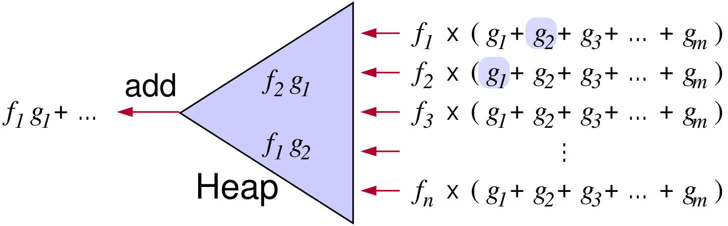
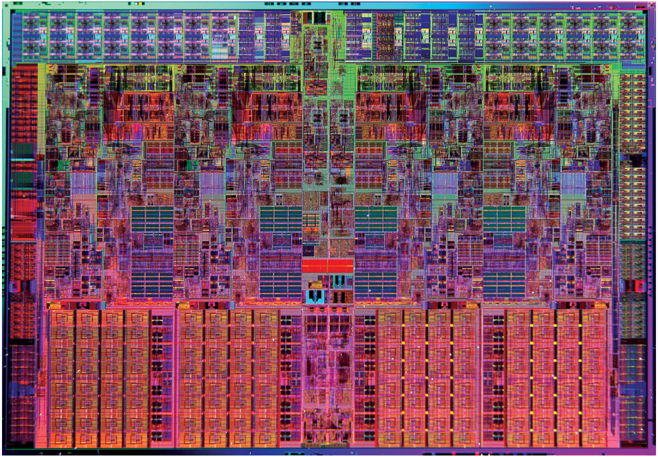


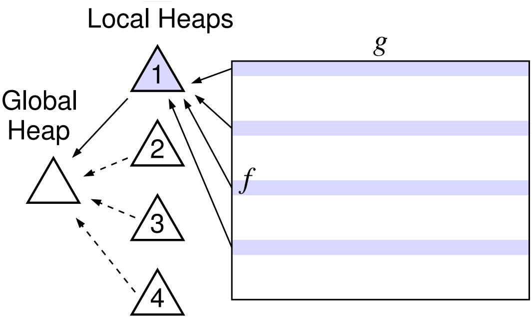
- $|Heap| \leq n \implies O(nm \log n)$ comparisons.
- Can pick $n \leq m$.
- Algorithm outputs $f \times g$ in descending order.

Target Parallel Architecture



Intel Core i7, quad core, shared memory.

Parallel Multiplication Algorithm



One heap per core.
Add (merge) results in global heap.

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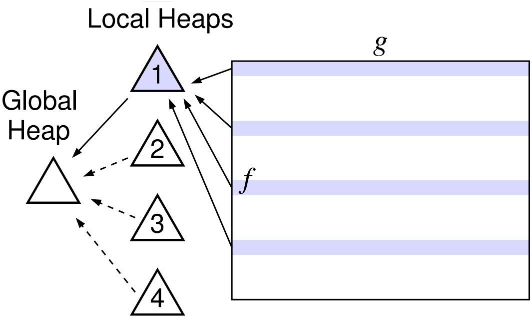

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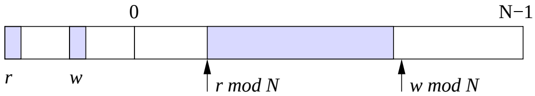
Add (merge) results
in global heap.

Threads write to a finite circular buffer.



Threads try to acquire global heap as buffer fills up to balance load.

Old multiplication and factorization benchmark.

Intel Core i5 750 2.66 GHz (4 cores) Times in seconds

multiply	Maple 13	Maple 16 1 core	Maple 16 4 cores	Magma 2.16-8	Singular 3.1.0	Mathematica 7
$p_1 := f_1(f_1+1)$	1.60	0.053	0.029	0.30	0.58	4.79
$p_3 := f_3(f_3+1)$	26.76	0.422	0.167	4.09	6.96	50.36
$p_4 := f_4(f_4+1)$	95.97	1.810	0.632	13.25	30.64	273.01
factor	Hensel lifting is mostly polynomial multiplication!!					
p_1 12341 terms	31.10	2.58	2.46	6.15	12.28	11.82
p_3 38711 terms	391.44	15.19	13.00	117.53	97.10	164.50
p_4 135751 terms	2953.54	53.52	44.84	332.86	404.86	655.49

$f_1 = (1+x+y+z)^{20}+1$ 1771 terms

$f_3 = (1+x+y+z)^{30}+1$ 5456 terms

$f_4 = (1+x+y+z+t)^{20}+1$ 10626 terms

The Maple timings are for `expand(f1*(f1+1))` and `factor(p1)`.

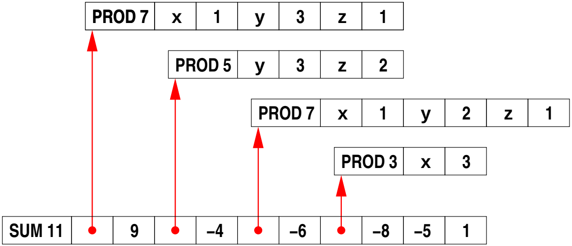
Maple Integration

To expand sums $f \times g$ Maple calls `` `expand/bigprod(f,g)` ``
if $\#f > 2$ and $\#g > 2$ and $\#f \times \#g > 1500$.

```
`expand/bigprod` := proc(a,b)  # multiply two large sums
   if type(a,polynom(integer)) and type(b,polynom(integer)) then
     x := indets(a) union indets(b); k := nops(x);
     A := sdmp:-Import(a, plex(op(x)), pack=k);
     B := sdmp:-Import(b, plex(op(x)), pack=k);
     C := sdmp:-Multiply(A,B);
     return sdmp:-Export(C);
   else
   ...

`expand/bigdiv` := proc(a,b,q)  # divide two large sums
   ...
     x := indets(a) union indets(b); k := nops(x)+1;
     A := sdmp:-Import(a, grlex(op(x)), pack=k);
     B := sdmp:-Import(b, grlex(op(x)), pack=k);
   ...
```

Almost everything is slow.



Many operations cost $O(nt)$. [n = #variables, t = #terms]
E.g. `indets(f); degree(f,x); coeff(f,x,i);`

Some operations add sorting cost of $O(t^{1.25})$.
E.g. `diff(f,x); expand(x*f); taylor(f,x,d);`

Make POLY the default representation in Maple.

If we can pack all monomials into one word use

SEQ 4	**x**	**y**	**z**

POLY 12		**5131**	**9**	**5032**	**–4**	**4121**	**–6**	**3300**	**–8**	**0000**	**–5**

$O(1)$	`degree(f); lcoeff(f); indets(f);`
$O(n)$	`has(f,z); type(f,polynom(integer));`
$O(n+t)$	`degree(f,x); expand(x*t); diff(f,x);`

For f with t terms in n variables.

Almost everything is fast.

command	Maple 16	Maple 17	speedup	notes
$\texttt{coeff}(f, x, 20)$	2.140 s	0.005 s	420x	terms easy to locate
$\texttt{coeffs}(f, x)$	0.979 s	0.119 s	8x	reorder exponents and radix
$\texttt{frontend}(g, [f])$	3.730 s	0.000 s	$\rightarrow O(n)$	looks at variables only
$\texttt{degree}(f, x)$	0.073 s	0.003 s	24x	stop early using monomial de
$\texttt{diff}(f, x)$	0.956 s	0.031 s	30x	terms remain sorted
$\texttt{eval}(f, x = 6)$	3.760 s	0.175 s	21x	use Horner form recursively
$\texttt{expand}(2 * x * f)$	1.190 s	0.066 s	18x	terms remain sorted
$\texttt{indets}(f)$	0.060 s	0.000 s	$\rightarrow O(1)$	first word in dag
$\texttt{subs}(x = y, f)$	1.160 s	0.076 s	15x	combine exponents, sort, me
$\texttt{taylor}(f, x, 50)$	0.668 s	0.055 s	12x	get coefficients in one pass
$\texttt{type}(f, polynom)$	0.029 s	0.000 s	$\rightarrow O(n)$	type check variables only

For f with $n = 3$ variables and $t = 10^6$ terms created by

```
f := expand(mul(randpoly(v,degree=100,dense),v=[x,y,z])):
```

New multiplication and factorization benchmark.

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More benchmarks and details available in preprint.

Profile for factor(p1);

Profile for `factor(p1);` Real time from 2.63s to 1.11s real.

function	#calls	Maple 16 time	Maple 16 time%	New Maple time	New Maple time%
coeftayl	216	0.999s	36.96	0.270s	22.39
expand	1934	0.561s	20.75	0.375s	31.09
factor/diophant	236	0.475s	17.57	0.371s	30.76
divide	419	0.267s	9.88	0.055s	4.56
factor	1	0.206s	7.62	0.017s	1.41
factor/hensel	1	0.140s	5.18	0.075s	6.22
factor/unifactor	2	0.055s	2.03	0.043s	3.57
total:	2809	2.703s	100.00%	1.206s	100.00%

The `coeftayl(f,x=a,k);` command is defined by `coeff(taylor(f,x=a,k+1),x,k);` and is computed via `eval(diff(f,x$k),x=a) / k!` which is 4x faster.

Notes on the new integration for Maple 17.

Let $f \in R[x_1, x_2, ..., x_n]$ with $\deg_{x_i} f > 0$. We store f using POLY if

(i) f has integer coefficients

(ii) $d > 1$ and $t > 1$ where $d = \deg f$ and $t = \#$terms.

(iii) we can pack all monomials of f into one 64 bit word, i.e. if $d < 2^b$ where $b = \lfloor \frac{64}{n+1} \rfloor$

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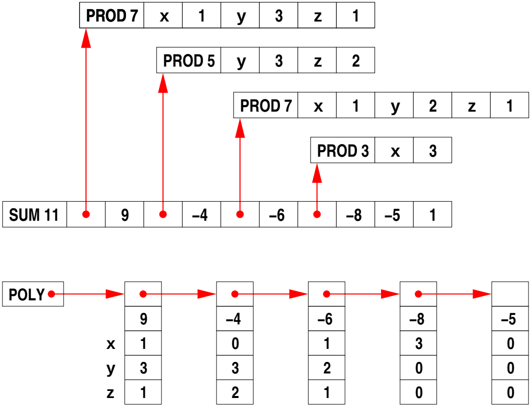
- Packing is fixed by $n = \#$variables.
- If $n = 8$, (iii) $\implies$ we use $b = \lfloor 64/9 \rfloor = 7$ bits per exponent field hence POLY restricts $d < 128$.
- The representation is invisible to the Maple user. Conversions are automatic.
- POLY polynomials will be displayed in sorted order.

Conclusion

We will not get good parallel speedup using these



Thank you for attending my talk.