In-place Arithmetic for Univariate Polynomials over an Algebraic Number Field

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Joint work with Mahdi Javadi.

GCDs over algebraic number fields.

Let $L = \mathbb{Q}(\alpha_1, \alpha_2, ..., \alpha_k)$.
Let f_1, f_2 be non-zero in $L[x_1, x_2, ..., x_n]$.
Compute $g = gcd(f_1, f_2)$.

Modular GCD Algorithms.

[1979] $k = 0, n \geq 1$: Zippel (CRT and sparse interpolation)

[1987] $k = 1, n = 1$: Langemyr and McCallum (CRT)

[1989] $k = 1, n = 1$: Geddes, Gonnet and Smedley

$$\mathbb{Z}[z]/m(z) \xrightarrow{z=a\in\mathbb{Z}} \mathbb{Z}_{m(a)}$$

[1995] $k = 1, n = 1$: Encarnacion (CRT + ratrecon)

[2002] $k \geq 1, n = 1$: van Hoeij and MM (CRT + ratrecon)

[2004] $k \geq 1, n \geq 1$: van Hoeij and MM (dense)

[2007] $k \geq 1, n \geq 1$: Javadi and MM (sparse)

[2007] $k \geq 1, n = 1$: Moreno Maza and Schost (FFT based)

Outline

- Example of GCD in $L[x]$ in Magma for $L = \mathbb{Q}(\sqrt[3]{2}, \sqrt{1 + \sqrt[3]{2}})$
- Using a recursive dense representation.
- In-place algorithms for $L_p[x]$.
- A benchmark with Magma for arithmetic $L_p[x]$.
- What's in the paper and current work.

Magma example.

```
Magma V2.15-15    Tue Nov 17 2009 12:25:02 on maple    [Seed = 475287942]
Type ? for help.  Type <Ctrl>-D to quit.

> Q := RationalField();
> P1<z1> := PolynomialRing(Q);
> m1 := z1^3-2; L1<z1> := quo<P1|m1>;     // L1 = Q[z1]/(z1^3-2)
> P2<z2> := PolynomialRing(L1);
> m2 := z2^2-1-z1; L2<z2> := quo<P2|m2>;  // L2 = L1[z2]/(z2^2-1-z1)
> IsField(L2);
true
> L<x> := PolynomialRing(L2);
> f1 := (x-z1-z2+2/3)*(x^2+z1*z2*x-1);
> f2 := (x-z1-z2+2/3)*((z2+z1^2+z1+6)*x-1);
> f1;
x^3 + ((z1-1)*z2 - z1 + 2/3)*x^2 + ((-z1^2+2/3*z1)*z2 -
    z1^2 - z1 - 1)*x + z2 + z1 - 2/3
> f2;
(z2 + (z1^2+z1+6))*x^2 + ((-z1^2-2*z1-16/3)*z2 -
    1/3*z1^2 - 19/3*z1)*x + z2 + z1 - 2/3
> Gcd(f1,f2);
x - z2 - z1 + 2/3
```

Magma example continued ...

The monic Euclidean algorithm.

```
> u := LeadingCoefficient(f2); u;
z2 + z1^2 + z1 + 6
> f2 := (1/u)*f2; // make f2 monic
> r1 := f1 mod f2;
> r1;

((-121/12675*z1^2 + 2362/12675*z1 - 964/12675)*z2 - 542/12675*z1^2 -
    226/12675*z1 - 11828/12675)*x + (-5702/38025*z1^2 + 638/2925*z1 +
    34282/38025)*z2 - 7129/38025*z1^2 + 30838/38025*z1 - 16786/38025

> u := LeadingCoefficient(r1); r1 := (1/u)*r1; // make r1 monic
> f2 mod r1;
0
> r1; // = gcd(f1,f2)
x - z2 - z1 + 2/3
```

Magma example continued ...

Using the modular algorithm of Monagan and van Hoeij (2002).

```
> p := 13;
> Zp := FiniteField(p);
> P1<z1> := PolynomialRing(Zp);
> m1 := z1^3-2; L1<z1> := quo<P1|m1>;
> P2<z2> := PolynomialRing(L1);
> m2 := z2^2-1-z1; L2<z2> := quo<P2|m2>;
> L<x> := PolynomialRing(L2);
> f1 := (x-z1-z2+2/3)*(x^2+z1*z2*x-1);
> f2 := (x-z1-z2+2/3)*((z2+z1^2+z1+6)*x-1);
> u := LeadingCoefficient(f2); u;
z2 + z1^2 + z1 + 6
> 1/u;

Runtime error in '/': Argument is not invertible

> IsField(L2);
false
```

Magma example continued ...

```
> p := 17;

> Zp := FiniteField(p);
...
> f2 := (x-z1-z2+2/3)*((z2+z1^2+z1+6)*x-1);
> u := LeadingCoefficient(f2);

> 1/u;
(13*z1^2 + 15*z1 + 14)*z2 + 12*z1^2 + 6*z1 + 13

> f2 := (1/u)*f2; // make f2 monic
> r1 := f1 mod f2;
> u := LeadingCoefficient(r1); r1 := (1/u)*r1; // make r1 monic
> f2 mod r1;
0
> g17 := r1; g17; // = gcd(f1,f2) mod 17
x + 16*z2 + 16*z1 + 12
```

Magma example continued ...

Repeat for additional primes.

```
> p := 19;
...
> g19 := r1; g19; // = gcd(f1,f2) mod 19
x + 18*z2 + 18*z1 + 7
```

Apply Chinese remaindering and rational reconstuction.

```
Maple 13 (X86 64 LINUX)
> g17 := x + 16*z2 + 16*z1 + 12:
> g19 := x + 18*z2 + 18*z1 + 7:
> g := iratrecon( chrem([g17,g19],[17,19]), 17*19 );

                    g := 2/3 + x - z2 - z1
```

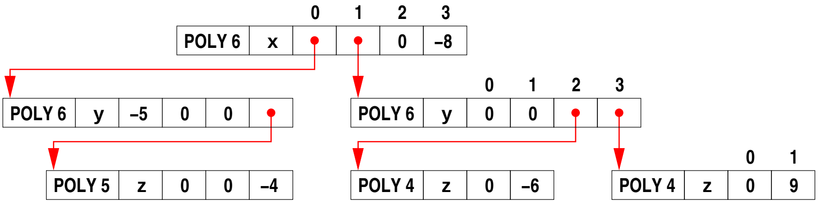
If g divides f_1 and f_2 stop and output g.

If we represent elements of $L = \mathbb{Q}(\alpha_1, \alpha_2, ..., \alpha_k)$ as polynomials in $\mathbb{Q}[z_1][z_2]\cdots[z_k] \mod \langle m_1(z_1), m_2(z_2), \ldots, m_k(z_k)\rangle$,

How do we do arithmetic in $L[x]$ mod p ?
How do we represent elements of $L[x]$ mod p ?

Recursive Dense Representations for Polynomials.

Pari's recursive dense representation for $\mathbb{Z}[z][y][x]$.

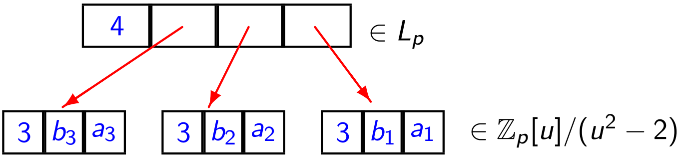


$$(-5y - 4z^2y^3) + (-6zy^2 + 9zy^3)x - 8x^3$$

Let $L = \mathbb{Q}(\sqrt{2}, \sqrt{5+\sqrt{2}})$.

Build $L_p[x] = \mathbb{Z}_p[u]/(u^2-2)[v]/(v^2-5-u)[x]$.

Use arrays of arrays of arrays of machine integers for $\mathbb{Z}_p[u][v][x]$.



An element $(a_1 u + b_1)v^2 + (a_2 u + b_2)v + (a_3 u + b_3) \in L_v$.

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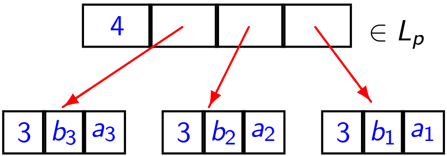

$\in L_p$

$\in \mathbb{Z}_p[u]/(u^2-2)$

An element $(a_1 u + b_1)v^2 + (a_2 u + b_2)v + (a_3 u + b_3) \in L_v$.

Consider $(au+b) \times (cu+d)$ in $L_u = \mathbb{Z}_p[u]/(u^2-2)$.

We multiply $(au+b) \times (cu+d) =$ | 4 | bd | $ad+bc$ | ac |

then reduce mod $u^2 - 2$ to get | 3 | $ad+bc$ | $bd+2ac$ |.

Real work: 6 multiplications in $\mathbb{Z}_p$ and 3 additions in $\mathbb{Z}_p$.

Main idea: reuse an array W of working storage to reduce the number of storage allocations from

$$O(\deg_x(f_1) \times \deg_x(f_2) \times \deg_{z_2}(m_2)^2 \times \ldots \times \deg_{z_k}(m_k)^2) \text{ to } O(1).$$

Our recursive dense representation.

Let $\bar{m}_1 = z_1^3 + 3 = m_1 \bmod p$ and
$\bar{m}_2 = z_2^2 + (5z_1 + 4)z_2 + (7z_1^2 + 3z_1 + 6) = m_1 \bmod p$ and
$f = (3 + 4z_1) + (5 + 6z_1)z_2 + ((7 + 8z_1 + 9z_1^2) + 10z_2)x.$

Representation for $E = [\bar{m}_1 = m_1 \bmod p, \bar{m}_2 = m_2 \bmod p]$.

$E =$

2	2	6	3	7	1	4	5	0	0	1	0	0	3	3	0	0	1
$\bar{m}_2$													$\bar{m}_1$				

Representation for f in $L_p[x]$.

$A =$

1	1	1	3	4	0	1	5	6	0	1	2	7	8	9	0	10	0	0
	$(3 + 4z_1) + (5 + 6z_1)z_2$									$(7 + 8z_1 + 9z_1^2) + 10z_2$								

Our in-place C library

Let $L = \mathbb{Q}(\alpha_1, ..., \alpha_K)$ and $D = \deg(L)$.
Let S_K be the space to represent one element of L mod p.

We have $S_K = 1 + \deg(m_K) \cdot S_{K-1} \implies D < S_K < 2D.$

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IP_MUL(K, E, p, a, b, c, W); $|W| < 6S_K$

Multiplies $a \times b$ in $L[x]$ mod p. Answer is written into c.

IP_REM(K, E, p, a, b, W); $|W| < 6S_K$

Divides a by b in $L[x]$ mod p. Quotient and remainder are written into a.

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IP_INV(K, E, p, a, W); $|W| < 15S_K$

Inverts a in L mod p. Answer is written into a.

IP_GCD(K, E, p, a, b, W); $|W| < 17S_K$

Computes $\gcd(a, b)$ in $L[x]$ mod p.

Answer is written into either a or b (both are destroyed).

Let $L_k = \mathbb{Z}[z_k][z_{k-1}]\ldots[z_1]/\langle m_1, \ldots, m_k, p\rangle)$.
Let $a = \sum_{i=0}^{d_a} a_i x^i$ and $b = \sum_{i=0}^{d_b} b_i x^i$ where $a_i, b_i \in L_k[x]$.
Let $c = a \times b = \sum_{i=0}^{d_a+d_b} c_i x^i$. Compute

$$c_j = \left[\sum_{i=\max(0,j-d_b)}^{\min(j,d_a)} a_i \times b_{j-i} \right] \mod m_k(z_k)$$

where $a_i \times b_{j-i}$ is done in $L_{k-1}[z_k]$.

- Needs working storage for $2 \in O(1)$ elements of $L_{k-1}[z_k]$ of degree $2(\deg_{z_k}(m_k) - 1)$.
- Division in $L_k[x]$ can be similarly implemented.

Benchmark

Here $p = 3037000453$, $L = \mathbb{Q}(\alpha_1, \alpha_2)$, m_1 and m_2 have degrees d_1 and d_2 such that $d = d_1 \times d_2 = 60$. We choose three polynomials a, b, g of degree d_x in x with coefficients chosen from L_p at random.

d_1	d_2	d_x	IP_MUL	MAG_MUL	IP_REM	MAG_REM	IP_GCD	MAG_GCD
2	30	40	0.124	0.050	0.123	0.09	0.384	2.26
3	20	40	0.108	0.054	0.106	0.11	0.340	2.35
4	15	40	0.106	0.056	0.106	0.10	0.327	2.39
6	10	40	0.106	0.121	0.105	0.14	0.328	5.44
10	6	40	0.100	0.093	0.100	0.37	0.303	7.84
15	4	40	0.097	0.055	0.095	0.17	0.283	3.27
20	3	40	0.092	0.046	0.091	0.14	0.267	2.54
30	2	40	0.087	0.038	0.087	0.10	0.242	1.85
2	30	80	0.477	0.115	0.478	0.27	1.449	9.41
3	20	80	0.407	0.127	0.409	0.27	1.304	9.68
4	15	80	0.404	0.132	0.406	0.28	1.253	9.98
6	10	80	0.398	0.253	0.400	0.35	1.234	22.01
10	6	80	0.380	0.197	0.381	0.86	1.151	31.57
15	4	80	0.365	0.127	0.364	0.40	1.081	13.49
20	3	80	0.353	0.109	0.353	0.33	1.030	10.59
30	2	80	0.336	0.086	0.337	0.26	0.932	7.83

Table: Timings in CPU seconds on an AMD Opteron 254 CPU running at 2.8 GHz

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Integer division optimization.

Multiplication in $\mathbb{Z}_p[z]$.

```
M = p*p;
d_c = d_a+d_b;
for( k=0; k<=d_c; k++ ) {
   t = 0;
   for( i=max(0,k-d_b); i <= min(k,d_a); i++ )
   {
      // t = (t+A[i]*B[k-i]) % p;
      if( t<0 ); else t = t-M;
      t = t+A[i]*B[k-i];
   }
   t = t % p;
   if( t<0 ) t = t+p;
   C[k] = t;
}
```

This improved performance of IP_GCD by a factor of 5 to 6.

What's in the paper?

- Pseudo-code for IP_MUL, IP_REM, IP_INV, IP_GCD.
- Formulas for bounds for $|W|$.
- Website link for repository of C code and test problems.

Current and future work.

- We have integrated IP_MUL, IP_GCD into Maple 14.
- Will integrate IP_REM into Maple 15.
- Are experimenting with an FFT based multiplication.