Optimizing and and Parallelizing the Modular GCD Algorithm

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Let $\bar{A} = A/G$ and $\bar{B} = B/G$ be the cofactors.
Let $A = \sum_{i=0}^{da} a_i(x_2, ..., x_n) x_1^i$.
Let $B = \sum_{i=0}^{db} b_i(x_2, ..., x_n) x_1^i$.
Let $G = \sum_{i=0}^{dg} g_i(x_2, ..., x_n) x_1^i$.
Let $t = \max_{i=0}^{dg} \#terms\ g_i$.

Interpolate $g_i(x_2, ..., x_n)$ modulo p from $2t + \delta$ univariate images in $\mathbb{Z}_p[x_1]$ using smooth prime p.

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Let $A = \sum_{i=0}^{da} a_i(x_2, ..., x_n) x_1^i$. $\quad CA = GCD(a_i(x_2, ..., x_n))$.
Let $B = \sum_{i=0}^{db} b_i(x_2, ..., x_n) x_1^i$. $\quad CB = GCD(b_i(x_2, ..., x_n))$.
Let $G = \sum_{i=0}^{dg} g_i(x_2, ..., x_n) x_1^i$. $\quad CG = GCD(CA, CB)$.
Let $t = \max_{i=0}^{dg} \#terms\ g_i$. $\quad \Gamma = GCD(a_{da}, b_{db})$.

Observation: Most of the time is recursive GCDs in $n-1$ variables and evaluation and interpolation not GCD in $\mathbb{Z}_p[x_1]$.

Bivariate Images

Compute $G = \mathrm{GCD}(A, B)$ in $\mathbb{Z}[x_1, x_2, ..., x_n]$.

Let $A = \sum_i a_{i,j}(x_3, ..., x_n)x_1^i x_2^j$. $CA = GCD(a_i(x_3, ..., x_n))$.
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1. Optimize serial bivariate Gcd computation.
2. For $n > 2$ parallelized (Cilk C) evaluation and interpolation.
3. Benchmark against Maple and Magma.

Bivariate Gcd computation.

Input $A, B \in \mathbb{Z}_p[y][x]$. Output $G = GCD(A, B)$, $\bar{A}$ and $\bar{B}$.

Trial division method. (Maple, Magma)
Interpolate y in G from univariate images in $\mathbb{Z}_p[x]$ **incrementally** until $G(x, y)$ does not change.
Test if $G|A$ and $G|B$. If yes output $G, \bar{A} = A/G, \bar{B} = B/G$.

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Cofactor recovery method. (Brown 1971)

Interpolate y in $G, \bar{A}, \bar{B}$ from univariate images
$g_i = G(\alpha_i, x), \bar{a}_i = A(\alpha_i, x)/g_i, \bar{b}_i = B(\alpha_i, x)/g_i$ in $\mathbb{Z}_p[x]$.
After k images we have

$$A - G\bar{A} \equiv 0 \pmod{M} \text{ and } B - G\bar{B} \equiv 0 \pmod{M}$$

where $M = (y - \alpha_1)(y - \alpha_2) \cdots (y - \alpha_k)$.
Stop when $k > \max(\deg_y A, \deg_y B, \deg_y G\bar{A}, \deg_y G\bar{B})$.

Cofactor recovery method for $\mathbb{Z}_p[y][x]$

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in batches until one of $G, \bar{A}, \bar{B}$ **stabilizes**.

Case G stabilizes: obtain remaining images using univariate $\div$
$g_i = G(\alpha_i, x), \bar{a}_i = A(\alpha_i, x)/g_i, \bar{b}_i = B(\alpha_i, x)/g_i$
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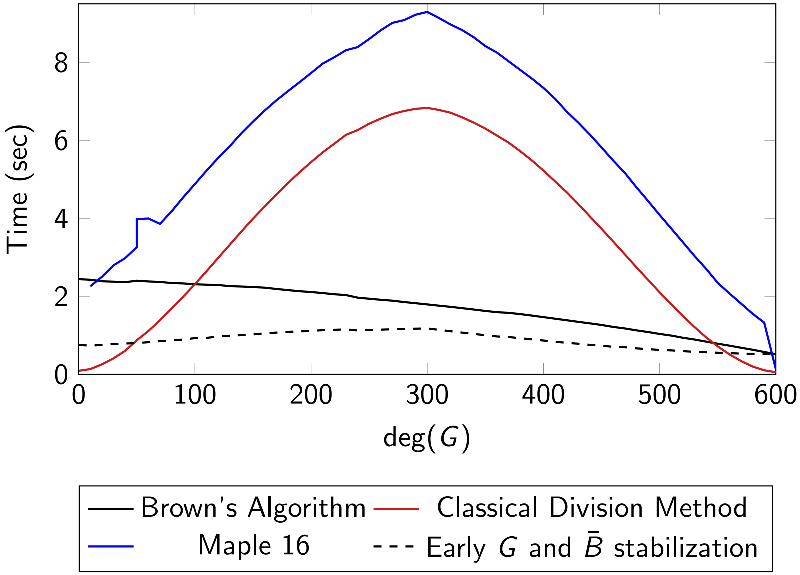
Case $\bar{A}$ stabilizes: obtain remaining images using univariate $\div$
$\bar{a}_i = \bar{A}(\alpha_i, x), g_i = A(\alpha_i, x)/\bar{a}_i, \bar{b}_i = B(\alpha_i, x)/g_i$
thus replacing the Euclidean algorithm with an evaluation.

Figure : Image Division Optimizations



Using FFT with small roots of unity

For dense A, B in $\mathbb{Z}_p[x_n][x_1 \dots x_{n-1}]$ we evaluate and interpolate A and B in blocks of size j using a FFT of size j ($j = 2, 4, 8, 16, \dots$).
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Parallel experiments in Cilk C

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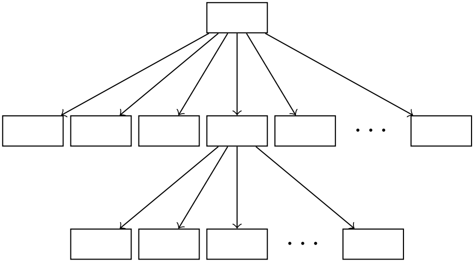
Cilk is a `C/C++` extension for parallelism in computation. **Cilk** uses a fixed number of worker threads and a work-stealing algorithm, and two basic keywords: `cilk_spawn` and `cilk_sync`.
We implement with Cilk Plus by Intel.

Dense Polynomial Structure Recursive dense representation using arrays. Multivariate polynomials form a tree.
A, B in $\mathbb{Z}_p[x_1, x_2, x_3]$, monic, dense in total degree $d = 200$



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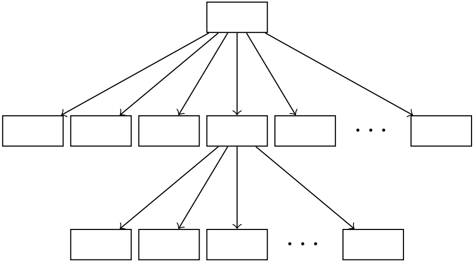

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$\mathbb{Z}_p[x_2, x_3]$ $\quad d + 1 = 201$

$\mathbb{Z}_p[x_3]$ $\quad \frac{d^2+3d+2}{2} = 20503$

The number of terms in each input polynomial is 1.37 million, filling 10.5 MB of memory.

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The algorithm is recursive and needs a lot of pieces of memory.
Many calls to `malloc` can be a bad idea.
We allocate large blocks of memory and use it as a stack.
Memory for each bivariate Gcd is all preallocated.

Benchmarks $A, B \in \mathbb{Z}_p[x_1, x_2, x_3]$, $\deg A = \deg B = 200$.

Table : Real times in seconds, $p = 2^{62} - 57$, inputs have 1373701 terms

$\deg(G)$	$\deg(\bar{A})$	−opt,fft	−fft	1	8	16	20	Conv
10	190	15.81	8.79	4.79	0.84	0.54	0.48	0.37
40	160	14.59	9.42	5.79	0.92	0.55	0.49	0.27
70	130	13.25	9.74	6.47	0.99	0.56	0.49	0.21
100	100	11.80	9.87	6.72	1.00	0.57	0.50	0.18
130	70	10.25	8.19	5.29	0.80	0.46	0.40	0.18
160	40	8.56	7.14	4.16	0.66	0.39	0.34	0.20
190	10	6.80	6.58	3.44	0.58	0.37	0.33	0.25

jude 2 x E5-2680 v2 CPUs, 10 cores, 2.8 GHz (3.6 GHz turbo).
Maximum theoretical speed-up on 20 cores: 15.56

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Deg		Maple		MagmaR		MGCD, #CPUs				POLY
G	$\bar{A}$	$A \times B$	GCD	$A \times B$	GCD	1	4	8	16	Conv
10	190	2.22	70.98	77.22	33.34	6.35	1.83	1.06	0.71	0.47
40	160	25.65	267.16	920.48	159.71	7.75	2.13	1.18	0.75	0.35
70	130	25.62	439.80	1624.6	462.09	8.72	2.35	1.27	0.75	0.28
100	100	25.43	453.27	1526.2	900.65	9.11	2.43	1.32	0.79	0.24
130	70	25.69	436.11	1559.2	14254.	7.11	1.92	1.04	0.62	0.23
160	40	25.44	282.04	934.45	7084.3	5.63	1.52	0.83	0.51	0.26
190	10	2.23	77.28	90.30	2229.8	4.69	1.29	0.74	0.47	0.32

gaby two E5-2660 CPUs, 8 cores at 2.2 GHz (3.0 GHz turbo).
Maximum theoretical speed-up on 16 cores: 11.73

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