Sparse Polynomial Multiplication and Division in Maple 14

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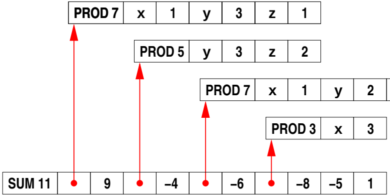
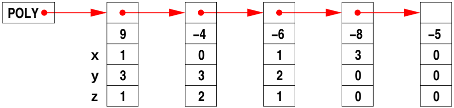
Polynomial Data Structures

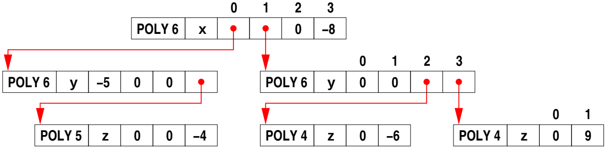


Maple



Singular

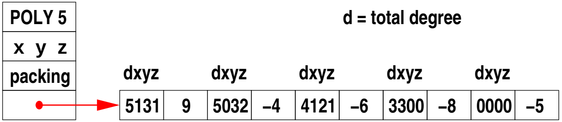
$$9xy^3z - 4y^3z^2 - 6xy^2z - 8x^3 - 5$$



Pari

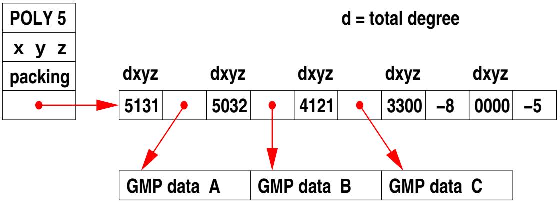
$$(-5y - 4z^2y^3) + (-6zy^2 + 9zy^3)x - 8x^3$$

Our Packed Array Representation



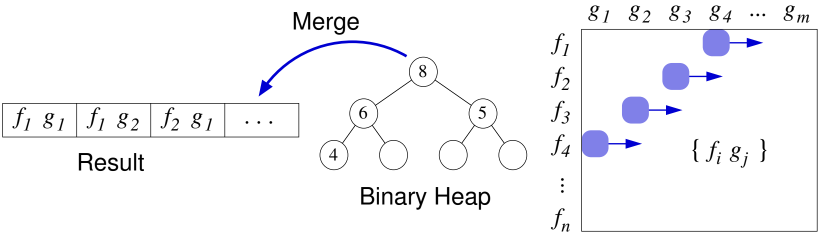
$$9xy^3z - 4y^3z^2 - 6xy^2z - 8x^3 - 5$$

encode $x^i y^j z^k$ as $[(i + j + k), i, j, k]$ in a 64 bit int (grlex)



Multiplication and Division Using a Heap

$f \cdot g = \sum f_i \cdot g$ simultaneous n-ary merge



multiply in $O(\#f \#g \log(\min(\#f, \#g)))$ comparisons

Monagan & Pearce: $f - \sum_{i=1}^{\#g} q_i \cdot g - \sum_{i=2}^{\#g} g_i \cdot (q_{\#g+1} + \dots)$

- $f \div g = q$ in $O(\#f + \#q\#g \log(\min(\#q, \#g)))$ comparisons
- $O(\#f + \#q\#g)$ multiplications, $O(\#q)$ divisions & gcds in $\mathbb{Z}$

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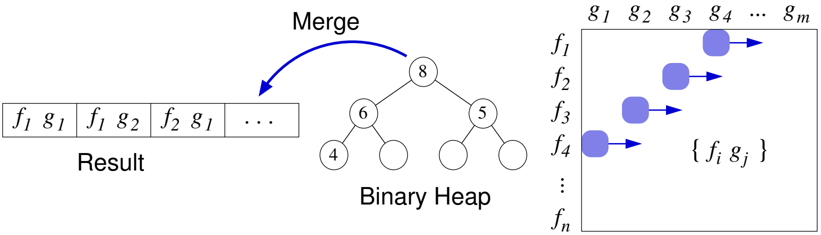

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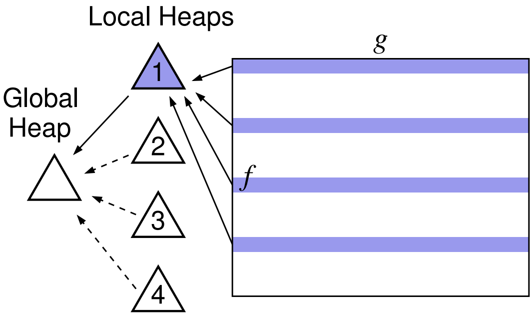
Parallel Algorithms



- distribute partial products $f_i \cdot g$ to cores
- threads take turns merging results in global heap
- superlinear speedup in practice (Core i3/i5/i7 & Phenom)
- parallel multiplication (ISSAC '09) now in Maple 14
- parallel division (PASCO '10) recently completed

Maple Integration

```
if #f > 2 and #g > 2 and (#f #g > 1500) then

‘expand/bigprod‘ := proc(a,b)  # multiply two large sums
   ...  # type check, variables, degree
   A := sdmp:-Import(a, plex(vars), pack=k);
   B := sdmp:-Import(b, plex(vars), pack=k);
   C := sdmp:-Multiply(A,B);
   return sdmp:-Export(C);

‘expand/bigdiv‘ := proc(a,b,q)  # exact division over Z
   ...  # type check, variables, degree
   A := sdmp:-Import(a, grlex(vars), pack=k);
   B := sdmp:-Import(b, grlex(vars), pack=k);
   if sdmp:-Divide(A,B,’Q’) then
     q := sdmp:-Export(Q);
     return true;
   else
     return false;
```

Benchmarks

Intel Core i7 920 2.66 GHz (4 cores, no hyperthreading) (64 bit Linux)
See paper for Singular, Mathematica, and Pari timings.

	Maple 13	Maple 14 1 core	Maple 14 4 cores		Magma 2.16-8	Trip 1.0 (RS)	Trip 1.0 (RD)
multiply $p_3 := f_3(f_3+1)$	26.76s	0.532s	0.166s	28%	4.09s	0.376s	0.314s
divide $q_3 := p_3/f_3$	24.74s	0.73s	0.74s	5%	4.31s	1.880s	1.722s
factor $\#p_3 = 38711$	391.44s	17.15s	15.59s		117.53s		

$f_1 = (1+x+y+z)^{20}+1$
1771 terms

$\mathbf{f_3 = (1+x+y+z)^{30}+1}$
5456 terms

$f_2 = (1+x^2+y^2+z^2)^{20}+1$
1771 terms

$f_4 = (1+x+y+z+t)^{20}+1$
10626 terms