# Towards High Performance Multivariate Factorization

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This is joint work with Baris Tuncer.

## The Multivariate Diophantine Problem (MDP)

Given $A, B, C \in \mathbb{Z}_p[x_1, x_2, \ldots, x_n]$ with $\gcd(A, B) = 1$ solve

$$\sigma A + \tau B = C$$

for $\sigma, \tau \in \mathbb{Z}_p[x_1, \ldots, x_n]$ s.t. $\deg_{x_1} \sigma < \deg_{x_1} B$.

## Outline

- Context: factoring $a \in \mathbb{Z}[x_1, \ldots, x_n]$ using Hensel lifting
- Wang's solution
- Kaltofen's sparse solution v. our sparse solution
- Eliminating multi-precision arithmetic
- Eliminating the error computation

# Multivariate polynomial factorization

**Input** $a = fg \in \mathbb{Z}[x_1, \ldots, x_n]$ primitive and square-free.
**Output** $f, g$.

1. Pick $x_1$ and factor $LC(a) = h_1^{e_1} h_2^{e_2} \ldots$ in $\mathbb{Z}[x_2, \ldots, x_n]$.
2. Pick $\alpha = \alpha_2, \ldots, \alpha_n \in \mathbb{Z}$ such that $LC(a)(\alpha) \neq 0$ and $\gcd(a(\alpha), \frac{\partial a}{\partial x_1}(\alpha)) = 1$ and $h_i(\alpha) \in \mathbb{Z}$ have distinct prime divisors.
3. Factor $a(x_1, \alpha) = f(x_1, \alpha)\ g(x_1, \alpha)$ over $\mathbb{Z}$.
4. Distribute the factors of $LC(a)$ on $f(x_1, \alpha)$ and $g(x_1, \alpha)$.
5. Pick a prime $p > \alpha_i$ and $L \in \mathbb{N}$ s.t. $p^L > 2\|f\|$ where $f|a$ and $\gcd(f(x_1, \alpha), g(x_1, \alpha)) = 1$ in $\mathbb{Z}_p[x_1]$.
   Hensel lift $x_2$ then $x_3$ then ... then $x_n$ doing arithmetic mod $p^L$.

P. Wang. An improved Multivariate Polynomial Factoring Algorithm.
*Mathematics of Computation*, **32**:1215–1231. AMS (1978)

*Algorithms for Computer Algebra*, K.O. Geddes, S.R. Czapor, G. Labahn. Kluwer, 1992.

# Wang's Multivariate Hensel Lifting (MHL)

**Input** $a \in \mathbb{Z}[x_1, ..., x_n]$, $\alpha = (\alpha_2, \ldots \alpha_n)$, $(f_0, g_0) \in \mathbb{Z}[x_1]$ s.t.

(i) $a(x_1, \alpha) - f_0 g_0 = 0$ and (ii) $\gcd(f_0, g_0) = 1$ in $\mathbb{Z}_p[x_1]$.

**Output** $f, g$ satisfying $a = fg$ or FAIL.

1. If $n = 1$ output $(f_0, g_0)$.
2. Lift $x_2, \ldots, x_{n-1}$: $(f_0, g_0) :=$ MHL( $a(x_n = \alpha_n)$, $\alpha$, $(f_0, g_0)$ )

   Idea: $f = \sum_{k=0}^{df} f_k (x_n - \alpha_n)^k$ and $g = \sum_{k=0}^{dg} g_k (x_n - \alpha_n)^k$.

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Idea: $f = \sum_{k=0}^{df} f_k (x_n - \alpha_n)^k$ and $g = \sum_{k=0}^{dg} g_k (x_n - \alpha_n)^k$.

3 Initialize: $(f, g) := (f_0, g_0)$; $error := a - fg$.

4 For $k = 1, \ldots, deg_{x_1} a$ while $error \neq 0$ do

$T_k :=$ Taylor coeff( $error$, $(x_n - \alpha_n)^k$ )
If $T_k \neq 0$ then

Solve $f_k \ g_0 + g_k \ f_0 = T_k$ for $f_k, g_k \in \mathbb{Z}_p[x_1, \ldots, x_{n-1}]$.
Set $f := f + f_k (x_n - \alpha_n)^k$ and $g := g + g_k (x_n - \alpha_n)^k$
Set $error := a - fg$

5 If $error = 0$ output $(f, g)$ else output FAIL.

# Wang's Multivariate Diophantine Lifting (MDP)

Input $A, B, C \in \mathbb{Z}_p[x_1, \ldots, x_n]$, $\alpha = \alpha_2, \ldots \alpha_n$ )

Output $\sigma, \tau \in \mathbb{Z}_p[x_1, \ldots, x_n]$ satisfying $\sigma A + \tau B = C$

1 If $n = 1$ solve $\sigma A + \tau B = C$ using the Euclidean algorithm in $\mathbb{Z}_p[x_1]$.

Let $\sigma = \sum_{k=0}^{df} \sigma_k (x_n - \alpha_n)^k$ and $\tau = \sum_{k=0}^{dg} \tau_k (x_n - \alpha_n)^k$.

2 $(\sigma_0, \tau_0) :=$ MultiDioLift( $A(x_n = \alpha_n)$, $B(x_n - \alpha_n)$, $C(x_n = \alpha_n)$, $\alpha$ )

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3 Initialize: $(\sigma, \tau) := (\sigma_0, \tau_0)$ $\quad$ $error := C - \sigma A - \tau B$

4 For $k = 1, 2, \ldots$ while $error \neq 0$ do

$\quad T_k :=$ Taylor coeff( $error$, $(x_n - \alpha_n)^k$ )

$\quad$ If $T_k \neq 0$ then

$\quad\quad \sigma_k, \tau_k :=$ MultiDioLift( $\sigma_0, \tau_0, T_k, \alpha$ )

$\quad\quad \sigma, \tau := \sigma + \sigma_k (x_n - \alpha_n)^k, \tau + \tau_k (x_n - \alpha_n)^k)$

$\quad\quad error := error - \sigma_k (x_n - \alpha_n)^k A - \tau_k (x_n - \alpha_n)^k B$

5 output $(\sigma, \tau)$.

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If $T_k \neq 0$ then

$\sigma_k, \tau_k :=$ MultiDioLift( $\sigma_0, \tau_0, T_k, \alpha$ )

$\sigma, \tau := \sigma + \sigma_k (x_n - \alpha_n)^k, \tau + \tau_k (x_n - \alpha_n)^k)$

$error := error - \sigma_k (x_n - \alpha_n)^k A - \tau_k (x_n - \alpha_n)^k B$

5 output $(\sigma, \tau)$.

Let $M(n)$ count calls to the Euclidean algorithm and $d \geq \max(deg_{x_i} f, \deg_{x_i} g)$.

Then $M(1) = 1, M(n) \leq dM(n-1) \Longrightarrow M(n) \leq d^{n-1}$.

Recall that $f = \sum_{i=0}^{df} f_i (x_n - \alpha_n)^i$ and $g = \sum_{i=0}^{dg} g_i (x_n - \alpha_n)^i$ .
Solve the MDP $f_k\ g_0 + g_k\ f_0 = T_k$ for $f_k, g_k \in \mathbb{Z}_p[x_1, \ldots, x_{n-1}]$.

Interpolate $x_2, ..., x_{n-1}$ from images in $\mathbb{Z}_p[x_1]$ using sparse interpolation ?
We tried Zippel's variable at a time sparse interpolation from 1979.

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**Theorem [MT]** The Strong Sparse Hensel Assumption
If $\alpha_n$ is random then, with high probability

$$\mathrm{supp}(f_0) \supseteq \mathrm{supp}(f_1) \supseteq \cdots \supseteq \mathrm{supp}(f_{df}) \text{ and}$$
$$\mathrm{supp}(g_0) \supseteq \mathrm{supp}(g_1) \supseteq \cdots \supseteq \mathrm{supp}(g_{dg}).$$

Example ($n = 8$, $\#f = 10,000$, $df = 16$, $\alpha = 3$) $|f_i| =$ 9877, 7043, 4932, 3374, 2310, 1545, 1001, 654, 418, 245, 141, 81, 34, 13, 5, 2, 1.

Recall that $f = \sum_{i=0}^{df} f_i(x_n - \alpha_n)^i$ and $g = \sum_{i=0}^{dg} g_i(x_n - \alpha_n)^i$ .
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**Kaltofen's Sparse Hensel Lifting.**
Let $\mathrm{supp}(f_0) = \{M_1, \ldots, M_r)$ and $\mathrm{supp}(g_0) = \{N_1, \ldots, N_s\}$ in $(x_1, \ldots, x_{n-1})$.
Equate the coefficients $c_i$ of

$$\left(\sum_{i=1}^{r} c_i M_i\right) g_0 + \left(\sum_{i=1}^{s} c_{r+i} N_i\right) f_0 = T_k \quad \text{in} \;\; \mathbb{Z}_p[x_1, \ldots, x_{n-1}]$$

in monomials in $(x_1, \ldots, x_{n-1})$ and solve the $(r+s) \times (r+s)$ linear system.

# MTSHL : Monagan and Tuncer's Sparse Hensel Lifting

Solve $f_k \; g_0 + g_k \; f_0 = T_k$ for $f_k, g_k \in \mathbb{Z}_p[x_1, \ldots, x_{n-1}]$.

Let $f_{k-1} = \sum_{i=1}^{df} a_i(x_2, \ldots, x_{n-1}) x_1^i$ and $\mathrm{supp}(a_i) = \{M_{i1}, \ldots, M_{ir_i}\}$,
and $g_{k-1} = \sum_{i=1}^{dg} b_i(x_2, \ldots, x_{n-1}) x_1^i$ and $\mathrm{supp}(b_i) = \{N_{i1}, \ldots, N_{is_i}\}$.
**Assume** $f_k = \sum_{i=0}^{df} \left( \sum_{j=1}^{r_i} a_{ij} M_{ij} \right) x_1^i$ and $g_k = \sum_{i=0}^{dg} \left( \sum_{j=1}^{s_i} b_{ij} N_{ij} \right) x_1^i$.

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Let $t = \max(\#a_i, \#b_i) \ll \#a + \#b$ . Let $\beta = (\beta_2, \ldots, \beta_{n-1}) \in \mathbb{Z}_p^{n-2}$.

Solve $\sigma_j\ g_0(\beta^j) + \tau_j\ f_0(\beta^j) = T_k(\beta^j)$ for $\sigma_j, \tau_j \in \mathbb{Z}_p[x_1]$. **for** $1 \le j \le t$.

Solve $df + dg$ Vandermonde systems

$$\{coeff(f_k(\beta^j), x_n^i) = coeff(\sigma_j, x_n^i) \text{ for } 1 \le j \le t\}$$
$$\{coeff(g_k(\beta^j), x_n^i) = coeff(\tau, x_n^i) \text{ for } 1 \le j \le t\}$$

Cost of MTSHL: number of arithmetic operations in $\mathbb{Z}_p$ where $d = max \deg_{x_i} a$

$$(n-1)d\ O(\underbrace{\#f\#g}_{error} + \underbrace{\#a}_{Taylor} + \underbrace{(\#a + \#f + \#g)\,t}_{eval} + \underbrace{d^2 t}_{UniDio} + \underbrace{dt^2}_{Vandermonde}).$$

# MTSHL with division

$f_k\ g_0 + g_k\ f_0 = T_k \implies g_k = (T_k - f_k g_0)/f_0.$

Let $f_{k-1} = \sum_{i=1}^{df} a_i(x_2,\dots,x_{n-1})x_1^i$ and $\mathrm{supp}(a_i) = \{M_{i1},\dots,M_{ir_i}\}$,

**Assume** $f_k = \sum_{i=0}^{df}\left(\sum_{j=1}^{r_i} a_{ij}M_{ij}\right)x_1^i.$

Let $t = \max(\#a_i, \#b_i)$ and $\beta = (\beta_2,\dots,\beta_{n-1}) \in \mathbb{Z}_p^{n-2}$.

Solve $\sigma_j\ g_0(\beta^j) + \tau_j\ f_0(\beta^j) = T_k(\beta^j)$ for $\sigma_j,\tau_j \in \mathbb{Z}_p[x_1]$. **for** $1 \le j \le t$.

Solve $df + dg$ Vandermonde systems obtained by equating coefficients of $x_1$

$$f_k(\beta^j) = \sigma_j \text{ and } g_k(\beta^j) = \tau_j \text{ for } 1 \le j \le t$$

Finally compute $g_k := (T_k - f_k g_0)/f_0$ in $\mathbb{Z}_p[x_1,\dots,x_{n-1}]$.

Cost of MTSHL: number of arithmetic operations in $\mathbb{Z}_p$ where $d = max \deg_{x_i} a$

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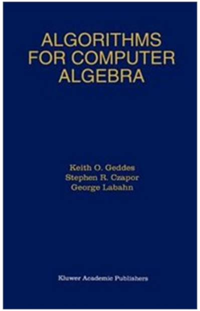
| $n/d/T$ | Wang (MDP) | Kaltofen (MDP) | MTSHL (MDP) |
|---|---|---|---|
| 4/35/100 | 13.07 (11.95) | 1.75 (1.18) | 1.51 (0.24) |
| 5/35/100 | 88.10 (86.28) | 3.75 (2.57) | 1.16 (0.36) |
| 7/35/100 | 800.0 (797.0) | 5.04 (4.08) | 1.58 (0.59) |
| 9/35/100 | 4451.6 (4449.4) | 8.13 (6.22) | 2.94 (0.56) |
| 4/35/500 | 33.96 (26.48) | 642.2 (635.1) | 11.29 (0.82) |
| 5/35/500 | 472.1 (402.5) | 1916.2 (1899.6) | 26.0 (4.86) |
| 7/35/500 | 3870.5 (3842.2) | 2329.4 (2305.5) | 43.1 (6.84) |
| 9/35/500 | $> 60000$ | 3866.3 (3805.9) | 79.6 (9.71) |

$> \ M := product(\ x_i,\ i = 1..n\ );$

$> \ f := x_1^d + M\, randpoly(\ [x_1, \ldots, x_n],\ terms = T,\ degree = d\ );$




Chapter 6
Newton's Iteration and the Hensel Construction

Run the entire Hensel lifting mod $p^L$
where $p^L > 2\max(\|a\|, \|f\|, \|g\|)$.
Why?

To avoid the expression swell when solving
$\sigma g_0(\alpha) + \tau f_0(\alpha) = T_k(\alpha)$ in $\mathbb{Z}[x_1]$ in MDP.

**Lemma [Gelfond]** Let $a \in \mathbb{Z}[x_1, \ldots, x_n]$ with $d_i = \deg_{x_i} a$.
If $f | a$ then

$$\|f\| \leq \mathrm{e}^{d_1+d_2+\ldots d_n}\|a\| \quad \text{where} \quad \mathrm{e} = 2.718281828$$

Let $p$ be a prime and let $f = \sum_{k=0}^{hf} f_k p^k$ and $g = \sum_{k=0}^{hg} g_k p^k$.

**Example**

$$\begin{aligned} f &= 2x_1 + (5 + 0 \cdot p + 2p^2)x_2 + (7 + p)x_3 \\ &= \underbrace{(2x_1 + 5x_2 + 7x_3)}_{f_0} + \underbrace{3x_3}_{f_1} p + \underbrace{2x_2}_{f_2} p^2. \end{aligned}$$

$$\mathrm{supp}(f_0) = \{x_1, x_2, x_3\} \supseteq \mathrm{supp}(f_1) = \{x_3\} \not\supseteq \mathrm{supp}(f_2) = \{x_2\}.$$

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**Theorem** If $p$ is chosen at random from $[2^{b-1}, 2^b]$ for $b$ sufficiently large then

$$\mathrm{supp}(f_0) \supseteq \mathrm{supp}(f_1) \supseteq \mathrm{supp}(f_2) \supseteq \ldots \mathrm{supp}(f_{hf}) \text{ and}$$
$$\mathrm{supp}(g_0) \supseteq \mathrm{supp}(g_1) \supseteq \mathrm{supp}(g_2) \supseteq \ldots \mathrm{supp}(g_{hg}) \text{ whp.}$$

**Idea:** Run the entire Hensel lifting modulo a **machine** prime $p$ to obtain $f_0, g_0 \in \mathbb{Z}_p[x_1, \ldots, x_n]$ satisfying $a - f_0 g_0 \mod p = 0$. Now do a $p$-adic lift on $f_0, g_0$ to get $f, g$.

# Multivariate $p$-adic Lifting

**Input** $a \in \mathbb{Z}[x_1, ..., x_n]$, $(f_0, g_0) \in \mathbb{Z}_p[x_1, \ldots, x_n]$ satisfying $a - f_0 g_0 \mod p = 0$ prime $p$ and a lifting bound $L$ satisfying $p^L > \|f\|$ where $f|a$.

**Output** $f, g \in \mathbb{Z}[x_1, ..., x_n]$ satifying $a = fg$ or FAIL.

1 Initialize: $(f, g) := (f_0, g_0)$; $error := a - fg$;

2 For $k = 1, \ldots, L$ while $error \neq 0$ do

$T_k := (\frac{error}{p^k}) \mod p.$

If $T_k \neq 0$ then

Solve the MDP $f_k \; g_0 + g_k \; f_0 = T_k$ for $f_k, g_k \in \mathbb{Z}_p[x_1, \ldots, x_n]$.
Set $f_k := mods(\sigma, p)$ and $g_k := mods(\tau, p)$.
Set $f := f + f_k \, p^k$; $g := g + g_k \, p^k$; $error := a - fg$.

3 If $error = 0$ output $(f, g)$ else output FAIL.

The MDP is solved in $\mathbb{Z}_p[x_1, \ldots, x_n]$ where $p$ is a machine prime!

| $n/d/T$ | $m$ | $L$ | MTSHL mod $p^L$ | MTSHL mod $p$ + lift |
|---|---|---|---|---|
| 5/10/300 | 2 | 5 | 5.86 (5.10) | 0.197 + 0.241 |
| 5/10/300 | 4 | 9 | 6.92 (6.10) | 0.514 + 0.553 |
| 5/10/300 | 8 | 17 | 8.63 (7.596) | 0.477 + 2.076 |
| 5/10/1000 | 2 | 5 | 14.44 (12.82) | 0.870 + 1.332 |
| 5/10/1000 | 4 | 9 | 16.94 (15.18) | 0.921 + 2.632 |
| 5/10/1000 | 8 | 17 | 19.03 (17.22) | 0.873 + 4.032 |

Table: CPU timings (seconds) for $p$–adic lifting for $p = 2^{31} - 1$.

$$> f \; := \; x_1^d \; + randpoly(\; [x_1, \ldots, x_n],\; terms = T,\; degree = d\;, coeffs = rand(p^m)\;)\,;$$

# Eliminating the error multiplication $a - fg$

Let $f^{(k)} = \sum_{i=0}^{k-1} f_i (x_n - \alpha_n)^i$ and $g^{(k)} = \sum_{i=0}^{k-1} g_i (x_n - \alpha_n)^i$.

**Main Idea:** $error(\beta^j) = a(\beta^j) - f^{(k)}(\beta^j)\ g^{(k)}(\beta^j)$ in $\mathbb{Z}_p[x_1, x_n]$.

Compute $a(\beta^j)$ for $1 \le j \le t$ before main loop.
Compute $f_{k-1}(\beta^j)$ and $g_{k-1}(\beta^j)$ for $1 \le j \le t$.

Let $f^{(k)}(\beta^j) = \sum_{i=0}^{k-1} a_i (x_n - \alpha_n)^i$ *and* $g^{(k)}(\beta^j) = \sum_{i=0}^{k-1} b_i (x_n - \alpha_n)^i$. Then

$$T_k(\beta^j) = \underbrace{coeff(a(\beta^j), (x_n - \alpha)^k)}_{O(d^2)} - \underbrace{\sum_{i=1}^{k-1} b_i(x_1) c_{k-i}(x_1)}_{O(d^2)}.$$

$$(n-1)d\ O(\underbrace{\#f\#g}_{error} + \underbrace{\#a}_{Taylor} + \underbrace{d^2 t}_{Taylor} + \underbrace{(\#a/d + \#f + \#g)\,t}_{eval} + \underbrace{d^2 t}_{UniDio} + \underbrace{dt^2}_{Vandermonde}).$$

We eliminated the the error computation in $\mathbb{Z}_p[x_1, \ldots, x_{n-1}]$ !

# Factoring the determinants of Cyclic and Toeplitz matrices

Let $C_n$ denote the $n \times n$ cyclic matrix and let $T_n$ denote the $n \times n$ symmetric Toeplitz matrix below.

$$C_n = \begin{pmatrix} x_1 & x_2 & \dots & x_{n-1} & x_n \\ x_n & x_1 & \dots & x_{n-2} & x_{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ x_3 & x_4 & \dots & x_1 & x_2 \\ x_2 & x_3 & \dots & x_n & x_1 \end{pmatrix} \quad T_n = \begin{pmatrix} x_1 & x_2 & \cdots & x_{n-1} & x_n \\ x_2 & x_1 & \cdots & x_{n-2} & x_{n-1} \\ & \ddots & \ddots & \ddots & \\ x_{n-1} & x_{n-2} & \cdots & x_1 & x_2 \\ x_n & x_{n-1} & \cdots & x_2 & x_1 \end{pmatrix}$$

The determinants of $C_n$ and $T_n$ are homogeneous polynomials in $n$ variables.

Example

$$\det T_4 = \left({x_1}^2 - x_1x_2 - x_1x_4 - {x_2}^2 + 2\,x_2x_3 + x_2x_4 - {x_3}^2\right) \left({x_1}^2 + x_1x_2 + x_1x_4 - {x_2}^2 - 2\,x_2x_3 + x_2x_4 - {x_3}^2\right)$$

$$\det C_4 = (x_4 + x_3 + x_1 + x_2)\,(x_4 - x_3 - x_1 + x_2)\,\left({x_1}^2 - 2\,x_1x_3 + {x_2}^2 - 2\,x_2x_4 + {x_3}^2 + {x_4}^2\right)$$

| $n$ | $\#d_n$ | #factors | Maple | (MDP) | MTSHL | Magma | Singular |
|---|---|---|---|---|---|---|---|
| 7 | 427 | 30,56 | 0.035 | 30% | 0.046 | 0.01 | 0.02 |
| 8 | 1628 | 167,167 | 0.065 | 43% | 0.073 | 0.04 | 0.05 |
| 9 | 6090 | 153,294 | 0.166 | 73% | 0.122 | 0.10 | 0.28 |
| 10 | 23797 | 931,931 | 0.610 | 76% | 0.418 | 0.89 | 1.77 |
| 11 | 90296 | 849,1730 | 2.570 | 74% | 1.138 | 1.96 | 8.01 |
| 12 | 350726 | 5579,5579 | 19.45 | 80% | 13.16 | 72.17 | 84.04 |
| 13 | 1338076 | 4983,10611 | 84.08 | 84% | 21.77 | 181.0 | 607.99 |
| 14 | 5165957 | 34937,34937 | 637.8 | 77% | 249.9 | 6039.0 | 20333.45 |
| 15 | 19732508 | 30458,66684 | 4153.2 | 84% | 1651.7 | 12899.2 | – |

Table: Factorization timings (seconds) for det $T_n$ evaluated at $x_n = 1$

Notes: Intel Core i5-4670 CPU @ 3.40GHz, 16 gigs RAM.
Maple 17: kernelopts(numcpus=1), Magma 2.22–5, Singular 3–1–6,

| $n$ | $\#d_n$ | #fmax | Maple | (MDP) | MTSHL | Magma | Singular |
|---|---|---|---|---|---|---|---|
| 7 | 246 | 924 | 0.045 | 90% | 0.026 | 0.01 | 0.02 |
| 8 | 810 | 86 | 0.059 | 46% | 0.063 | 0.07 | 0.06 |
| 9 | 2704 | 1005 | 0.225 | 74% | 0.120 | 0.74 | 0.24 |
| 10 | 7492 | 715 | 0.853 | 62% | 0.500 | 8.44 | 2.02 |
| 11 | 32066 | 184756 | 7.160 | 91% | 0.945 | 104.3 | 11.39 |
| 12 | 86500 | 621 | 19.76 | 76% | 5.121 | 7575.1 | 30.27 |
| 13 | 400024 | 2704156 | 263.4 | 92% | 27.69 | 30871.9 | ?? |
| 14 | 1366500 | 27132 | 1664.4 | 77% | 523.07 | $> 10^6$ | 288463.2 |
| 15 | 4614524 | 303645 | 18432. | 82% | 7496.9 | – | – |

Table: Factorization timings (seconds) for det $C_n$ evaluated at $x_n = 1$

Notes: ?? = I cannot compute det$(C_n)$ nor read in det$(C_n)$ nor it's factors.