Ch 11 Rational Function Integration.

Let K be a field and $K(x) = \{ \frac{P(x)}{Q(x)} : P, Q \in K[x],\ Q \neq 0 \}$.
$K(x)$ is the set of rational functions in x over K.

Given $f(x) \in \mathbb{Q}(x)$ calculate $\int f(x)\,dx$.

E.g. $\int \frac{1}{x^2}dx = -\frac{1}{x} \in \mathbb{Q}(x)$ [Note: We will ignore the constant of $\int$]

$\int \frac{1}{x}dx = \ln x \notin \mathbb{Q}(x) \implies \mathbb{Q}(x)$ is not closed under $\int$

Theorem 11.4 $\int \frac{dx}{x} \notin \mathbb{Q}(x)$.

TAC suppose $\int \frac{dx}{x} \in \mathbb{Q}(x)$.

$\implies \boxed{\int \frac{dx}{x} = \frac{p(x)}{q(x)}}$ for some $p, q \in \mathbb{Q}[x]$, $q \neq 0$, $\boxed{\overset{\text{WLOG}}{\gcd(p,q)=1}}$

$\overset{'}{\implies} \frac{1}{x} = \frac{p'q - pq'}{q^2}$

$\implies \underset{\neq 0}{q^2} = \underset{\neq 0}{x}\underbrace{(p'q - pq')}_{\neq 0}$ in $\mathbb{Q}[x]$

x is irreducible so $x \mid q$. Let $q = x^n \bar{q}$ where $x \nmid \bar{q}$ and $\boxed{n>0}$

$\implies x^{2n}\bar{q}^2 = x(p'x^n\bar{q} - p[nx^{n-1}\bar{q} + x^n\bar{q}'])$

$\implies x^{2n}\bar{q}^2 = x^{n+1}p'\bar{q} - npx^n\bar{q} - x^{n+1}p\bar{q}'$

$\mid x^n \overset{}{\underset{n>0}{\implies}} x^n\bar{q}^2 = xp'\bar{q} - np\bar{q} - xp\bar{q}'$

$\implies x \mid np\bar{q} \implies x \mid p$. But $x \mid q$.

$\implies x \mid \gcd(p,q)$ $\boxtimes$

Therefore $\int \frac{1}{x}dx \notin \mathbb{Q}(x)$.

$\int \frac{x}{2x^2+1}dx = \int \frac{\frac{1}{2}x}{1 \cdot x^2 + 1/2}dx$.

Let $f(x) = \frac{P(x)}{Q(x)}$ where $P, Q \in K[x]$, $\gcd(P,Q) = 1$, $\deg Q > 0$,
and $\operatorname{lc}(Q) = 1$.

If $\deg P \geq \deg Q$ then $(P \div Q)$ $P = Qq + r$ for some quotient q and remainder r s.t. $r = 0$ or $\deg r < \deg Q$.

Then $$\int \frac{P}{Q}\,dx = \int \frac{Qq+r}{Q}\,dx = \underbrace{\int q\,dx}_{\text{polynomial part}} + \underbrace{\int \frac{r}{Q}\,dx}_{\text{proper rational function.}}$$

Method of partial fractions

Ex 1. $$\int \frac{x}{x^2-2}\,dx = \int \frac{\frac{1}{2}dx}{x-\sqrt{2}} + \int \frac{\frac{1}{2}dx}{x+\sqrt{2}} = \frac{1}{2}\ln(x-\sqrt{2}) + \frac{1}{2}\ln(x+\sqrt{2}) = \frac{1}{2}\ln(x^2-2)$$

($x \leftarrow P$, $x^2-2 \leftarrow Q$)

$$\frac{1}{2}(\ln A + \ln B) = \frac{1}{2}\ln AB$$

$$\frac{x}{x^2-2} = \frac{x}{(x-\sqrt{2})(x+\sqrt{2})} = \frac{A}{x-\sqrt{2}} + \frac{B}{x+\sqrt{2}} = \frac{\frac{1}{2}}{x-\sqrt{2}} + \frac{\frac{1}{2}}{x+\sqrt{2}}$$

$\times Q$ $\quad x = A(x+\sqrt{2}) + B(x-\sqrt{2})$

$x = \sqrt{2}$: $\quad \sqrt{2} = A(2\sqrt{2}) + B\cdot 0 \Rightarrow A = 1/2$

$x = -\sqrt{2}$ $\quad -\sqrt{2} = A\cdot 0 + B(-2\sqrt{2}) \Rightarrow B = 1/2$

We did arithmetic over $\mathbb{Q}(\sqrt{2})$.

Can we avoid factoring $Q(x)$ over $\mathbb{C}$? Sometimes.

Ex 2. $$\int \frac{2x^3-x^2+2}{x^2(x^2-2)}\,dx = \int \frac{-dx}{x^2} + \int \frac{2x\,dx}{x^2-2} = \underbrace{\frac{1}{x}}_{\text{rational part}} + \underbrace{1\cdot \ln(x^2-2)}_{\text{logarithmic part.}}$$

degree denominator -1.

$$\frac{2x^3-x^2+2}{x^2(x^2-2)} = \frac{Ax+B}{x^2} + \frac{Cx+D}{x^2-2} = \frac{-1}{x^2} + \frac{2x}{x^2-2}$$

relatively prime.

$\Rightarrow A=0, B=-1, C=2, D=0$

Can we extract the rational part of the $\int$ easily? Yes.

Theorem 11.6 Let $f(x) = P/Q$ where $P, Q \in K[x]$, K is a field. Suppose $\gcd(P,Q) = 1$, $0 \leq \deg P < \deg Q$ and $\operatorname{lc} Q = 1$.

Let $Q = q_1 q_2^2 q_3^3 \dots q_k^k$ where $\gcd(q_i, q_j) = 1$ and $\gcd(q_i, q_i') = $

Suppose $\gcd(P,Q)=1$, $0 \le \deg P < \deg Q$ and $\operatorname{lc} Q = 1$.

Let $Q = q_1 q_2^2 q_3^3 \cdots q_k^k$ where $\gcd(q_i, q_j)=1$ and $\gcd(q_i, q_i')=1$.

There exist $A, C \in K[x]$ such that

$$(1) \quad \int \frac{P}{Q} = \frac{A}{B} + \int \frac{C}{D} = \frac{A}{q_2 q_3^2 \cdots q_k^{k-1}} + \int \frac{C}{q_1 q_2 \cdots q_k}.$$

rational part

logarithmic part.

where $B = \gcd(Q, Q') = q_2^1 q_3^2 \cdots q_k^{k-1}$

and $D = Q/B = q_1' q_2' \cdots q_k$

and $\deg A < \deg B$ and $\deg C < \deg D$

Hermite
Horowitz

PFD
Trager/Rothstein.

Uniqueness? Impose $\operatorname{lc}(q_i) = 1 \Rightarrow A$ and C are unique.

Integration by Parts.

$$\boxed{\int uv = \left(\int u\right)\cdot v - \int \int u \cdot v'}$$

$$uv = u \cdot v + \int u \cdot v' - \int u \cdot v'$$

$$\int \underset{u}{x} \underset{v}{\ln x}\, dx$$

$$= \tfrac{1}{2}x^2 \cdot \ln x - \int \tfrac{1}{2}x^2 \cdot \tfrac{1}{x}\, dx$$

$$= \tfrac{1}{2}x^2 \ln x - \int \tfrac{1}{2} x\, dx$$

$$= \tfrac{1}{2}x^2 \ln x - \tfrac{1}{4}x^2 + C.$$

$$\int \underset{v}{x}\, \underset{u}{e^x}\, dx = e^x \cdot x - \int e^x \cdot 1 = xe^x - e^x.$$