Lec2A Big O notation
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To determine the cost of an algorithm let $T(n)$ be the number of ______ operations that the algorithm does for an input of size $n$.
If $T(n) = 3n^2 + 2n + 5$ we say the algorithm is quadratic in $n$.
For large $n$ the term $3n^2$ dominates the cost of the alg.

Definition: Let $n \in \mathbb{N}$ and $f: \mathbb{N} \to \mathbb{R}$ and $g: \mathbb{N} \to \mathbb{R}$.
We say g dominates f if $\exists\, c > 0$ and $\exists\, k \in \mathbb{N}$ such that $|f(n)| \le c\,|g(n)|$ for $n \ge k$.

Define $O(g(n)) = \{ f(n) : g(n) \text{ dominates } f(n) \}$.

Examples. $\qquad |f(n)| \le c\,|g(n)| \;\; \forall n \ge k$.

$2n^2+5 \in O(n^2)$ because $2n^2+5 \le \boxed{7}\,n^2 \;\; \forall n \ge \boxed{1}$

$3n+1 \in O(n^2)$ because $3n+1 \le 4n^2 \;\; \forall n \ge 1$

$n^3 \notin O(n^2)$ because $n^3 \not\le cn^2 \;\; \forall n > c$

So $O(n^2) = \{ an^2+bn+c,\ an+b,\ a\cdot n^{1.5}+bn+c,\ n\log_2 n, \dots \}$

$O(n) \subset O(n^2) \subset O(n^3)$

Linear algs. — Quadratic algorithms — Cubic algorithms.

e.g. $\mathbb{Z}$ $+, -$ length $n$ digits — $\mathbb{Z}$ $\times, \div$ — ? Gaussian elimination

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$$\begin{array}{c|c} & 1 \quad\cdots\quad n \\ \hline 1 & \begin{bmatrix} x & x & \cdots & x \\ \vdots & & & \\ x & x & \cdots & x \end{bmatrix} \\ n & \end{array}$$

How do we show $O(f(n)) = O(g(n))$? $\quad O(3n^2+11) = O(n^2+2n)$?
We show $O(f(n)) \subset O(g(n))$ and $O(g(n)) \subset O(f(n))$.

I claim if $f(n) \in O(g(n))$ then $\overset{h(n)}{O}(f(n)) \subset O(g(n))$.

Proof: Suppose $f(n) \in O(g(n)) \Rightarrow |f(n)| \le c_1 |g(n)| \;\; \forall n \ge k_1$.
Let $h(n) \in O(f(n)) \Rightarrow |h(n)| \le c_2 |f(n)| \;\; \forall n \ge k_2$
$\Rightarrow |h(n)| \le c_2 \cdot [c_1 |g(n)|] \;\; \forall n \ge \max(k_1, k_2)$.
$= c_1 \cdot c_2 |g(n)| \;\; \forall n \ge$ "

$\Rightarrow h(n) \in O(g(n))$.

Exercise: For $a>1$ and $b>1$ show $O(\log_a n) = O(\log_b n)$.
$= O(\log n)$

Hint: $\log_a n = \dfrac{\ln n}{\ln a}$.

Properties of $O(g(n))$.

① $O(c\,f(n)) = O(f(n))$ for any constant $c>0$.
E.g. $O(3n^2) = O(n^2)$. Proof: Exercise.

② If $f(n) \in O(g(n))$ then $O(|f(n)|+|g(n)|) = O(|g(n)|)$.
E.g. $O(\underbrace{3n^2}_{g(n)} + \underbrace{2n+11}_{f(n)}) \underset{②}{=} O(3n^2) \underset{①}{=} O(n^2)$.

③ $f(n)\cdot O(g(n)) = O(f(n)\cdot g(n))$ e.g. $2n\,O(n) = O(2n^2)$.

Often we want to add $O(f(n)) + O(g(n))$. For example

Algorithm foo

| | | | |
|---|---|---|---|
| Step 1 ........ | $\leq 2n\log_2 n + n$ | $\in$ | $O(n\log n)$ |
| Step 2 ------ | $= 3n^2+2n-1$ | $\in$ | $O(n^2)$ |
| Step 3 ------ | $= 2n+5$ | $\in$ | $O(n)$ |

$O(n\log n) + O(n^2) + O(n) = O(n\log n + n^2 + n) = O(n^2)$

Define $O(f(n)) + O(g(n)) = O(|f(n)| + |g(n)|)$

E.g. $O(1+n^2) + O(n^2) = O(1+2n^2) = O(n^2)$.

Suppose $f(x), g(x) \in \mathbb{Z}[x]$ of degree $n$.

Suppose we want to compute $h(x) = f(x)\cdot g(x) \bmod x^{n+1}$

E.g. $(1+2x)(2+3x) \bmod x^2 = 1\cdot 2 + (1\cdot 3 + 2\cdot 2)\cdot x^1$.
$n=1$ $\quad = 2 + 7x$

Let $f(x) = a_0 + a_1 x + \cdots + a_n x^n$

and $g(x) = b_0 + b_1 x + \cdots + b_n x^n$

Let $h(x) = c_0 + c_1 x^1 + \cdots + c_n x^n$

$c_0 = a_0 b_0$

$c_1 = a_0 b_1 + b_0 a_1$

$c_0 = a_0 b_0$
$c_1 = a_0 b_1 + b_0 a_1$
$c_2 = a_0 b_2 + a_1 b_1 + a_2 b_0$
$\vdots$
$c_n = a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0.$

Algorithm SeriesMult.

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for $k = 0, 1, \ldots, n$ do # compute $c_k$
    $c_k = 0$
    for $i = 0, 1, \ldots, k$ do
        $c_k = c_k + a_i \cdot b_{k-i}$ $\} \leftarrow 2(k+1) \in O(k).$

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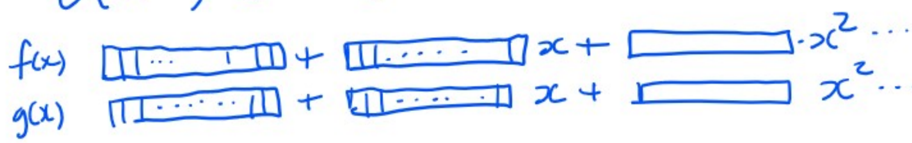
Let $T(n)$ be the # of arithmetic operations in $\mathbb{Z}$ that alg. SeriesMult does.

$$T(n) = \sum_{k=0}^{n} 2k+2 = \sum_{k=0}^{n} 2k + \sum_{k=0}^{n} 2$$
$$= 2\sum_{k=0}^{n} k + 2(n+1)$$
$$= \cancel{2}\frac{n(n+1)}{\cancel{2}} + 2(n+1) = (n+1)(n+2) \in O(n^2)$$

$$T(n) = \sum_{k=0}^{n} O(k) = O(0) + O(1) + O(2) + \cdots + O(n)$$
$$= O(0 + 1 + 2 + \cdots + n)$$
$$= O\left(\frac{n(n+1)}{2} = \frac{n^2}{2} + \frac{n}{2}\right) = O(n^2)$$ arithmetic operations in $\mathbb{Z}$.

Question: Is the time complexity of Algorithm SeriesMult $O(n^2)$? No.




The time will also depend on the cost of the coefficient arithmetic.