Lec5B Univariate Polynomials

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2.5 Univariate Polynomial Rings

Let R be a ring and $a \in R[x]$.

Let $a = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ with $a_n \neq 0$.

	Math	Maple
n is the degree of a w.r.t. x	$\deg a$	degree(a,x)
a_n is the leading coefficient	$\mathrm{lc}\, a$	lcoeff(a,x)
x^n is the leading monomial	$\mathrm{lm}\, a$	lcoeff(a,x,'m')
$a_n x^n$ is the leading term	$\mathrm{lt}\, a$	—

The zero polynomial.

NB $\mathrm{lc}(0) = \mathrm{lt}(0) = 0$, $\mathrm{lm}(0) = 1$, $\deg(0) = -\infty$

Theorem. If D is an integral domain and $a, b \in D[x]$ are non-zero polynomials then

(i) $\deg(ab) = \deg a + \deg b$.

(ii) $\mathrm{lc}(ab) = a_n \cdot b_m = \mathrm{lc}(a) \cdot \mathrm{lc}(b)$

(iii) $\mathrm{lm}(ab) = x^n \cdot x^m = \mathrm{lm}(a) \cdot \mathrm{lm}(b)$

(iv) $\mathrm{lt}(ab) = \mathrm{lt}(a)\,\mathrm{lt}(b)$

$$a \times b = (\overset{\neq 0}{a_n} x^n + \cdots + a_0)(\overset{\neq 0}{b_m} x^m + \cdots + b_0)$$
$$= \underbrace{a_n \cdot b_m}_{\neq 0} x^{n+m} + \cdots + a_0 \cdot b_0$$

$a \div b$.

Division in $F[x]$, $b \neq 0$ F a field.

Theorem. There exist unique polynomials $q, r \in F[x]$ s.t.

$a = bq + r$ and $r = 0$ or $\deg r < \deg b$.

Proof (uniqueness) Suppose

(1) $a = bq_1 + r_1$ where $r_1 = 0$ or $\deg r_1 < \deg b$

(2) $a = bq_2 + r_2$ where $r_2 = 0$ or $\deg r_2 < \deg b$.

(1)−(2) $0 = b(q_1 - q_2) + r_1 - r_2$

$\Rightarrow$ $b(q_1 - q_2) = r_2 - r_1$

$\Rightarrow$ $b \mid \underbrace{r_2 - r_1}_{\substack{\deg r_1 < \deg b \\ \deg r_2 < \deg b.}}$ $\Rightarrow$ $r_2 - r_1 = 0 \Rightarrow r_2 = r_1$

$0 = b \cdot (q_1 - q_2) + 0 \Rightarrow q_1 - q_2 = 0.$

$$0 = b\cdot(q_1 - q_2) + 0 \Rightarrow q_1 - q_2 = 0.$$

$b \neq 0$ — NO Z.D.S. $\Rightarrow q_1 = q_2.$

In $F[x]$

No Z.Ds.

Proof (existence) Division Algorithm.

$$\begin{array}{rl} & \quad 2x+2 \leftarrow q \\ b = 5x-3 & \big)\, 10x^2+4x+1 = a = r \\ 2x\cdot b & \; -(10x^2-6x) \\ & \qquad 10x+1 = r \\ 2\times b & \; -(10x-6) \\ & \qquad\quad 7 = r \end{array}$$

$a = bq + r.$

deg(r) deceases at least 1 at each ÷ step.

LOOP Invariant. $\boxed{a = bq + r}$

$a = b\cdot 0 + a$ ✓

$r \leftarrow a$

$q \leftarrow 0$

while $r \neq 0$ and $\deg r \geq \deg b$ do

$t \leftarrow lt(r)/lt(b)$

$q \leftarrow q + t$

$r \leftarrow r - t\cdot b$

end.

$a = bq + r$

$a = b(q_{old} + t) + (r_{old} - tb)$

$a = b\,q_{old} + r_{old}$ ✓

① Do we get here → return (q, r).

② Does $a = bq + r$?

③ Is $r = 0$ or $\deg r < \deg b$?