7.2 Trigonometric Integrals

Assignment # 3 due next Monday.

$\int \sin^m x \cdot \cos^n x \, dx$ $\quad m \geq 0, \; n \geq 0$

$\int \tan^m x \cdot \sec^n x$

$$\boxed{\int x^l \sin^m x \cos^n x \, dx \qquad l \geq 0, \; m \geq 0, \; n \geq 0}$$

$u = 2x$

$$\int \sin x \cdot \cos x \, dx = \int \frac{1}{2} \sin 2x \, dx = -\frac{1}{4} \cos 2x + C$$

$$\int \sin^2 x \, dx = \int \left(\frac{1}{2} - \frac{1}{2}\cos 2x\right) dx = \frac{1}{2}x - \frac{1}{4}\sin 2x + C$$

$$\int \cos^2 x \, dx = \int \left(\frac{1}{2} + \frac{1}{2}\cos 2x\right) dx = \frac{1}{2}x + \frac{1}{4}\sin 2x + C$$

Recall

$$\sin 2x = 2 \sin x \cos x \Rightarrow \boxed{\sin x \cdot \cos x = \frac{1}{2}\sin 2x} \quad ①$$

$$\cos 2x = 2\cos^2 x - 1 \Rightarrow \boxed{\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x} \quad ②$$

$$\sin^2 x + \cos^2 x = 1 \Rightarrow \sin^2 x = 1 - \cos^2 x \Rightarrow \boxed{\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x} \quad ③$$

Strategy for $\int \sin^m x \cos^n x \, dx$

CASE 1 n odd, $n \geq 3$ e.g. $\int \sin^2 x \cos^3 x \, dx$

Use $\cos^2 x = 1 - \sin^2 x$ to reduce n to 1. Then use $u = \sin x$.

E.g. $\int \sin^2 x \cos^3 x \, dx = \int \sin^2 x (1 - \sin^2 x) \cdot \cos x \, dx = \int (\sin^2 x - \sin^4 x) \underline{\cos x \, dx}$

Let $u = \sin x$

$du/dx = \cos x$

$du = \underline{\cos x \, dx}$

$= \int (u^2 - u^4) \, du = \frac{1}{3}u^3 - \frac{1}{5}u^5 + C$

$= \frac{1}{3}\sin^3 x - \frac{1}{5}\sin^5 x + C.$

CASE 2 m odd, $m \geq 3$ e.g. $\int \sin^3 x \cos^2 x \, dx.$

Use $\sin^2 x = \underline{1 - \cos^2 x}$ to reduce m to 1. Then use $u = \cos x$.

$\int \sin^3 x \cos^2 x \, dx = \int (1 - \cos^2 x) \cdot \cos^2 x \cdot \sin x \, dx = \int (\cos^2 x - \cos^4 x) \underline{\sin x \, dx}.$

$u = \cos x$

$\frac{du}{dx} = -\sin x \Rightarrow du = -\sin x \, dx$

$= \int -(u^2 - u^4) \, du$

$= -\frac{1}{3}u^3 + \frac{1}{5}u^5 + C$

$= -\frac{1}{3}\cos^3 x + \frac{1}{5}\cos^5 x + C.$

CASE 3 $\quad m$ and n both even e.g. $\int \sin^2 x \cdot \cos^2 x \, dx$

Use $\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x$ and $\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x$ repeatedly.

E.g. $\sin^2 x \cdot \cos^2 x = \left(\frac{1}{2} - \frac{1}{2}\cos 2x\right)\left(\frac{1}{2} + \frac{1}{2}\cos 2x\right) = \frac{1}{4} - \frac{1}{4}(\cos 2x)^2$

$$= \frac{1}{4} - \frac{1}{4}\left(\frac{1}{2} + \frac{1}{2}\cos 4x\right)$$

$$= 1/4 - 1/8 - \frac{1}{8}\cos 4x$$

$$= \underline{1/8 - 1/8 \cos 4x.}$$

$$\int \sin^2 x \cdot \cos^2 x \, dx = \int \left(\frac{1}{8} - \frac{1}{8}\cos 4x\right) dx$$

$$= \frac{1}{8}x - \frac{1}{32}\sin 4x + C$$

$\int \sin(mx) \cdot \cos(nx) \, dx$ $\quad$ Use $\sin A \cos B = \frac{1}{2}(\sin(A-B) + \sin(A+B))$ $\quad$ (1)

$\int \sin(mx) \cdot \sin(nx) \, dx$ $\quad$ Use $\sin A \sin B = \frac{1}{2}(\cos(A-B) - \cos(A+B))$ $\quad$ (2)

$\int \cos(mx) \cos(nx) \, dx$ $\quad$ Use $\cos A \cos B = \frac{1}{2}(\cos(A-B) + \cos(A+B))$ $\quad$ (3)

E.g. $\int \sin \underbrace{4x}_{A} \cdot \cos \underbrace{3x}_{B} \, dx \overset{(1)}{=} \int \frac{1}{2}(\sin x + \sin 7x) \, dx$

$$= -\frac{1}{2}\cos x - \frac{1}{14}\cos 7x + C$$

This method (using ①, ② and ③) + $\int$ by parts works for $\int x^{\ell} \cdot \sin^m x \cos^n x \, dx$, $\ell \geq 0$.

E.g. $\int x \boxed{\sin x \cos x} \, dx = \int x \, \frac{1}{2} (\sin 0 + \sin 2x) \, dx = \frac{1}{2} \int \underline{x \sin 2x \, dx}$

$\sin A \cos B = \frac{1}{2} (\sin(A-B) + \sin(A+B))$

$$\int f \cdot g' = f \cdot g - \int g \cdot f'$$

$$\int \underset{f}{x} \, \underset{g'}{\sin 2x} \, dx = x \left(-\tfrac{1}{2} \cos 2x\right) - \int \left(-\tfrac{1}{2} \cos 2x\right) \cdot 1 \, dx = -\tfrac{1}{2} x \cos 2x + \int \tfrac{1}{2} \cos 2x \, dx$$

$$= -\tfrac{1}{2} x \cos 2x + \tfrac{1}{4} \sin 2x + C.$$

$g = -\frac{1}{2} \cos 2x$

$\int \tan^m x \sec^n x \, dx$

1 $\int \tan x \, dx = \int \frac{\sin x}{\cos x} dx = \int -\frac{\sin x}{u} \cdot \frac{du}{\sin x} = \int -\frac{1}{u} du = -\ln|u| + C$

$= \underline{-\ln|\cos x| + C}$

$u = \cos x$

$du/dx = -\sin x \quad dx = \frac{-du}{\sin x}$

$-\ln A = \ln \frac{1}{A}$

$A > 0$

$= \ln\left|\frac{1}{\cos x}\right| + C$

$= \underline{\ln|\sec x| + C}$

2 $\int \sec x \, dx = \ln|\sec x + \tan x| + C.$

$(\ln|\sec x + \tan x|)' = \frac{\tan x \cdot \sec x + \sec^2 x}{\sec x + \tan x} = \frac{\sec x \cancel{(\tan x + \sec x)}}{\cancel{\sec x + \tan x}} =$

$\sec x = \frac{1}{\cos x} = (\cos x)^{-1}$

$\Rightarrow \boxed{\sec' x =} -1 \cdot (\cos x)^{-2}(-\sin x) = \frac{\sin x}{\cos^2 x} = \boxed{\tan x \cdot \sec x}$

3. $\int \tan x \cdot \sec x \, dx = \sec x + C$

4 $\int \sec^2 x\, dx = \tan x \, dx$

5 $\int \tan^2 x\, dx = \int (\sec^2 x - 1)\, dx = \int \sec^2 x\, dx - \int 1\, dx$

$\sec^2 x = 1 + \tan^2 x$

$= \tan x - x + C.$

6. $\int \tan^2 x \cdot \underline{\sec^2 x\, dx} = \int u^2 \cdot du = \frac{1}{3}u^3 + C = \frac{1}{3}\tan^3 x + C.$

Let $u = \tan x$

$\frac{du}{dx} = \sec^2 x \Rightarrow du = \sec^2 dx$