7.3 Trigonometric Integrals — Some rough road ahead :)

	Integrands	Substitution	Identity
①	$\sqrt{a^2-x^2}$	$x=a\cdot\sin\theta$	✓ $\sin^2\theta+\cos^2\theta=1$
②	$\sqrt{a^2+x^2}$	$x=a\cdot\tan\theta$	? $\tan^2\theta+1=\sec^2\theta$
③	$\sqrt{x^2-a^2}$	$x=a\cdot\sec\theta$	$\sec^2\theta-1=\tan^2\theta$

$\div\cos^2\theta$

$$\int\sqrt{1-x^2}\,dx=\int\sqrt{1-\sin^2\theta}\,dx=\int\sqrt{\cos^2\theta}\,dx=\int\cos\theta\,\underline{\cos\theta\,d\theta}$$

① $a=1$ $\quad x=1\cdot\sin\theta$

$$\frac{dx}{d\theta}=\cos\theta\Rightarrow dx=\cos\theta\,d\theta$$

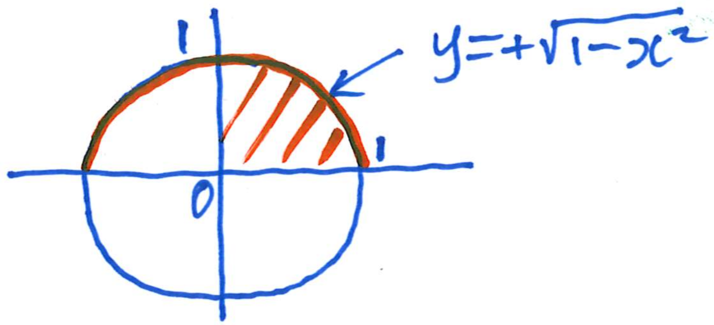
$$\int\sqrt{4+x^2}\,dx=\int\sqrt{4+4\tan^2\theta}\,dx=\int2\sqrt{1+\tan^2\theta}\,dx=\int2\sqrt{\sec^2\theta}\;2\sec^2\theta\,d\theta$$

$$=\int4\sec^3\theta\,d\theta.$$

② $a=2$ $\quad x=2\tan\theta$

$$\frac{dx}{d\theta}=2\cdot\sec^2\theta\Rightarrow dx=2\sec^2\theta\,d\theta$$

Example. Find the area of a circle of radius 1.



$x^2+y^2=1^2$

$y^2=1-x^2$

$y=\pm\sqrt{1-x^2}$

$$A=4\int_0^1\sqrt{1-x^2}\,dx=4\int_0^{\pi/2}\sqrt{1-\sin^2\theta}\,\cos\theta\,d\theta$$

Let $\boxed{x=\sin\theta}$

$\frac{dx}{d\theta}=\cos\theta$

$dx=\cos\theta\,d\theta$

$x=\boxed{1=\sin\theta}$

$\Rightarrow\ \theta=\pi/2$

$x=0=\sin\theta$

$\Rightarrow\ \theta=0.$

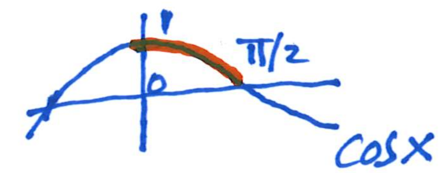
$$=4\int_0^{\pi/2}\cos^2\theta\,d\theta$$

$$=4\int_0^{\pi/2}\left(\frac{1}{2}+\frac{1}{2}\cos 2\theta\right)d\theta$$

$$=4\left[\frac{\theta}{2}+\frac{1}{4}\sin 2\theta\right]_0^{\pi/2}$$

$$=4\Big[\Big(\frac{\pi}{4}+\frac{1}{4}\underbrace{\sin\pi}_{0}\Big)-\Big(0+\frac{1}{4}\underbrace{\sin 0}_{0}\Big)\Big]$$

$$=\pi$$ ☺



From 7.2

Formula 30. $\int\sqrt{a^2-u^2}\,du=\frac{u}{2}\sqrt{a^2-u^2}+\frac{a^2}{2}\sin^{-1}\frac{u}{a}+C$

$a=1$

$$4\int_0^1\sqrt{1-x^2}\,dx=4\left[\frac{x}{2}\sqrt{1-x^2}+\frac{1}{2}\sin^{-1}x\right]_0^1$$

$$=4\big[(0+\pi/4)-(0+0)\big]=\pi.$$

Example 4 $\displaystyle\int_0^{1} \frac{dx}{\sqrt{1+x^2}} = \int_0^{\pi/4} \frac{\sec^2\theta}{\sqrt{1+\tan^2\theta = \sec^2\theta}}\, d\theta = \int_0^{\pi/4} \frac{\sec^2\theta}{\sec\theta}\, d\theta$

$= \displaystyle\int_0^{\pi/4} \sec\theta\, d\theta$

$= \Big[\ln|\sec\theta + \tan\theta|\Big]_0^{\pi/4}$

$= \ln|\sqrt{2}+1| - \ln|1+0|$

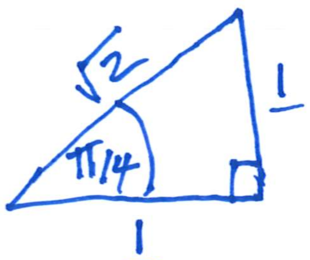
$= \ln(\sqrt{2}+1).$

② $\boxed{x = \tan\theta}$

$\frac{dx}{d\theta} = \sec^2\theta \Rightarrow dx = \sec^2\theta\, d\theta$

$x = 1 = \tan\theta \Rightarrow \theta = \pi/4$

$x = 0 = \tan\theta \Rightarrow \theta = 0$



$\tan \pi/4 = \frac{1}{1}$

$\cos \pi/4 = \frac{1}{\sqrt{2}}$

$\sec \pi/4 = 1/\cos \pi/4 = \sqrt{2}$

$\cos 0 = 1$

$\sec 0 = 1/1$

Formula 25 $\displaystyle\int \frac{du}{\sqrt{a^2+u^2}}\, du = \ln\left(u + \sqrt{a^2+u^2}\right) + C$

$\displaystyle\int_0^1 \frac{1}{\sqrt{1+x^2}}\, dx = \left[\ln\left(x + \sqrt{1+x^2}\right)\right]_0^1 = \ln(1+\sqrt{2}) - \ln(0+1) = \ln(1+\sqrt{2}).$

Example 5 $\displaystyle\int \frac{dx}{\sqrt{x^2-a^2}} = \int \frac{a\cdot\sec\theta\cdot\tan\theta}{\sqrt{a^2\sec^2\theta - a^2}}\,d\theta = \int \frac{a\sec\theta\tan\theta}{a\sqrt{\sec^2\theta-1}}\,d\theta$ $\quad \sec^2\theta - 1 = \tan^2\theta$

Let $\boxed{x = a\cdot\sec\theta}$

$\frac{dx}{d\theta} = a\cdot\sec\theta\cdot\tan\theta$

$dx = a\cdot\sec\theta\cdot\tan\theta\,d\theta$

$\boxed{\sec\theta = x/a}$

$\cos\theta = a/x$

$= \int \sec\theta\,d\theta = \ln|\sec\theta + \tan\theta| + C$

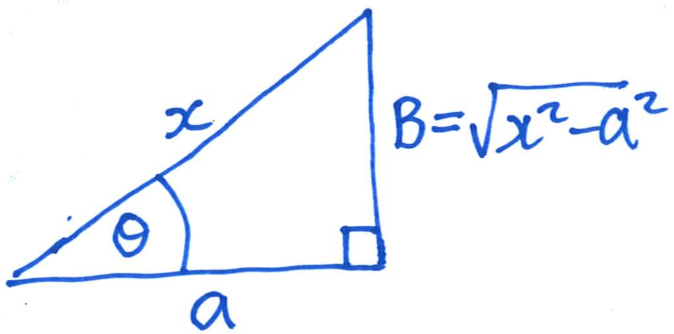
$= \ln\left|\frac{x}{a} + \frac{\sqrt{x^2-a^2}}{a}\right| + C$

$\boxed{\ln\frac{A}{B} = \ln A - \ln B}$ $\quad = \ln|x+\sqrt{x^2-a^2}| - \ln a + C$

$\boxed{\displaystyle\int \frac{dx}{\sqrt{x^2-a^2}}\,dx = \ln|x+\sqrt{x^2-a^2}| + C'}$

where $C' = C - \ln a$.

Formula 43



$\tan\theta = \frac{\sqrt{x^2-a^2}}{a}$

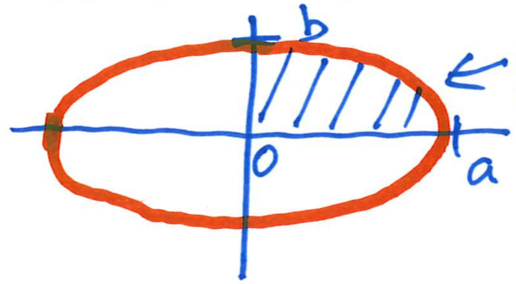
$a^2 + B^2 = x^2$

$\Rightarrow B^2 = x^2 - a^2$

$\Rightarrow B = \sqrt{x^2-a^2}$

Example 2 Find the area enclosed by the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$



$\leftarrow y=\frac{b}{a}\sqrt{a^2-x^2}$

$x=0 \quad y=b$

$y=0 \quad x=a$

$$\frac{y^2}{b^2}=1-\frac{x^2}{a^2}$$

$$y^2=b^2-\frac{b^2x^2}{a^2}=b^2\left(1-\frac{x^2}{a^2}\right)$$

$$y=\pm\frac{b}{a}\sqrt{a^2-x^2}$$

$$A=4\int_0^a \frac{b}{a}\sqrt{a^2-x^2}\,dx = 4\cdot\frac{b}{a}\int_0^a \sqrt{a^2-x^2}\,dx$$

Let $\boxed{x=a\sin\theta}$

$$\frac{dx}{d\theta}=a\cos\theta$$

$x=a=a\sin\theta$

$\Rightarrow \theta=\pi/2$

$x=0=a\sin\theta$

$\Rightarrow \theta=0$

$$=4\frac{b}{a}\int_0^{\pi/2}\underbrace{\sqrt{a^2-a^2\sin^2\theta}}_{a\sqrt{1-\sin^2\theta}}\cdot a\cos\theta\,d\theta$$

$$=4\frac{b}{a}\int_0^{\pi/2} a\cdot\cos\theta\cdot a\cos\theta\,d\theta$$

$$=4ab\,\underbrace{\boxed{\int_0^{\pi/2}\cos^2\theta\,d\theta}}_{=\pi/4}$$

$$=a\cdot b\cdot\pi$$