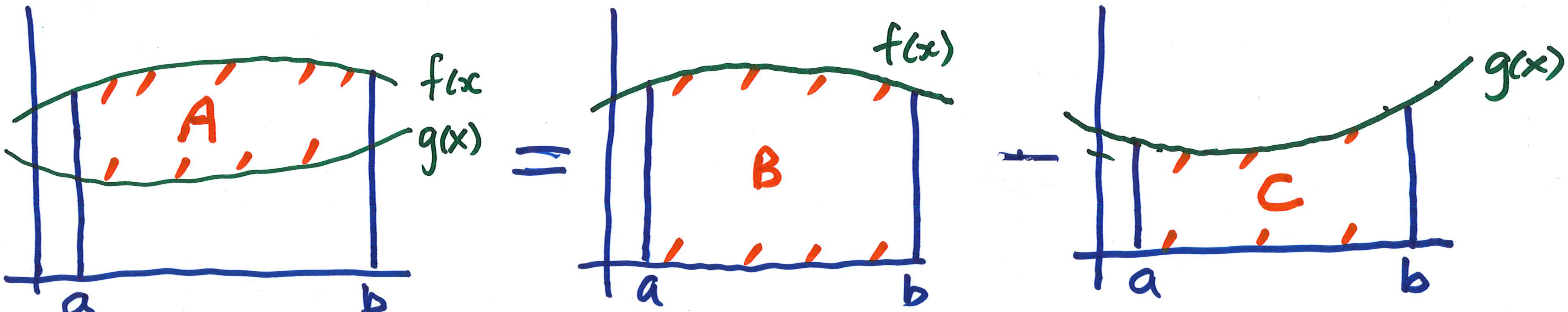
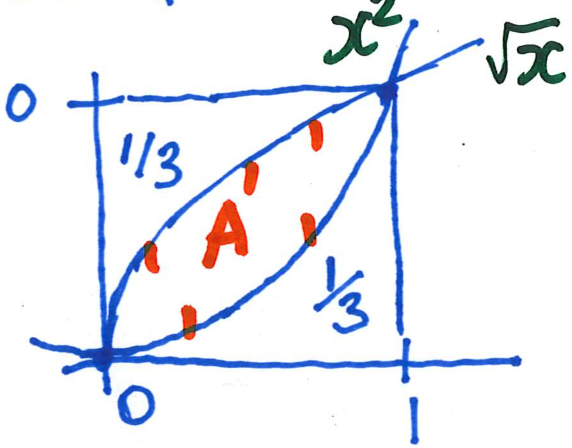
6.1 Areas Between Curves



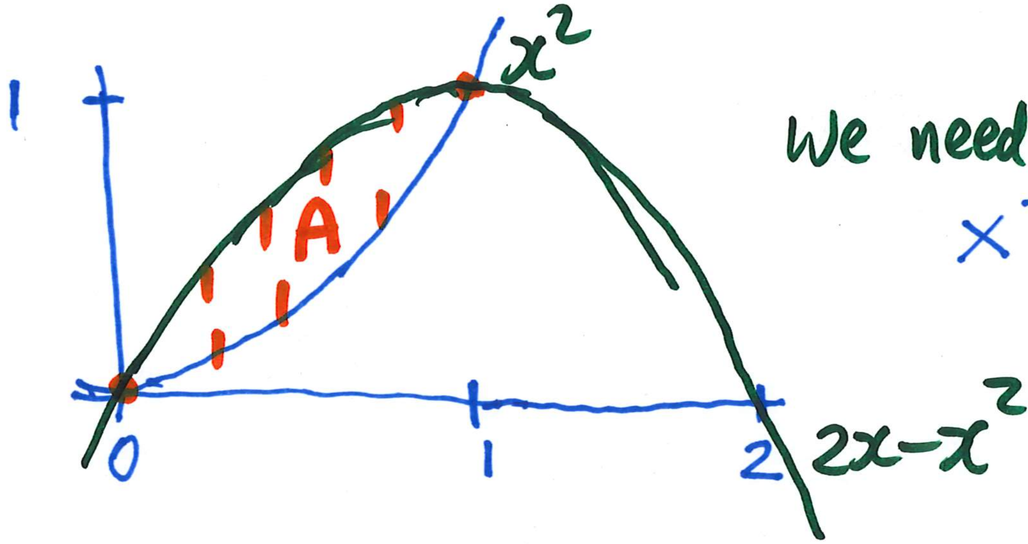
$$A = B - C = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx = \int_a^b (f(x) - g(x))\,dx$$

Example 1 What is the area between $\sqrt{x}$ and x^2 on $[0,1]$



$$A = \int_0^1 (\underset{\overset{\shortparallel}{x^{\frac{1}{2}}}}{\sqrt{x}} - x^2)\,dx = \left[\frac{2}{3}x^{\frac{3}{2}} - \frac{1}{3}x^3\right]_0^1 = \left(\frac{2}{3} - \frac{1}{3}\right) - 0 = \frac{1}{3}$$

Example 2 What is the area between $\underset{f(x)}{y=x^2}$ and $\underset{g(x)}{y=2x-x^2}$?



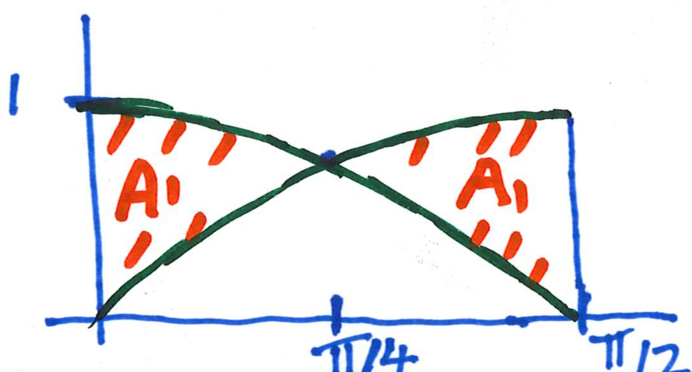
We need to solve $f(x)=g(x)$ to find their intersections.

$$x^2 = 2x - x^2 \Rightarrow x^2 - 2x + x^2 = 0$$
$$\Rightarrow 2x^2 - 2x = 0$$
$$\Rightarrow 2x(x-1) = 0$$
$$\Rightarrow x=0 \text{ or } x=1.$$

$$A = \int_0^1 \left((2x-x^2)-x^2\right) dx = \int_0^1 (2x - 2x^2)\,dx = \left[x^2 - \tfrac{2}{3}x^3\right]_0^1 = \left(1-\tfrac{2}{3}\right) - 0 = \tfrac{1}{3}$$

$$A = \int_0^1 \left(x^2 - (2x-x^2)\right) dx = \cdots\cdots = -\tfrac{1}{3}$$

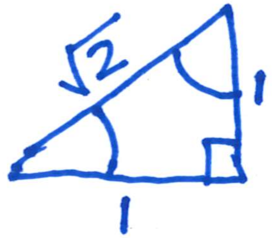
Example 3. Find the area bounded by $\sin x$, $\cos x$, $\underline{x=0,\ x=\frac{\pi}{2}}$



$$A = A_1 + A_2 = 2\int_0^{\pi/4} (\cos x - \sin x)\,dx$$
$$= 2\left[\sin x + \cos x\right]_0^{\pi/4} = 2\left(\sin\tfrac{\pi}{4} + \cos\tfrac{\pi}{4}\right)$$
$$- 2(\sin 0 + \cos 0)$$

$\pi/2 = 90^\circ$

$\pi/4 = 45^\circ$



$\sin \frac{\pi}{4} = \frac{1}{\sqrt{2}}$

$\cos \pi/4 = 1/\sqrt{2}$

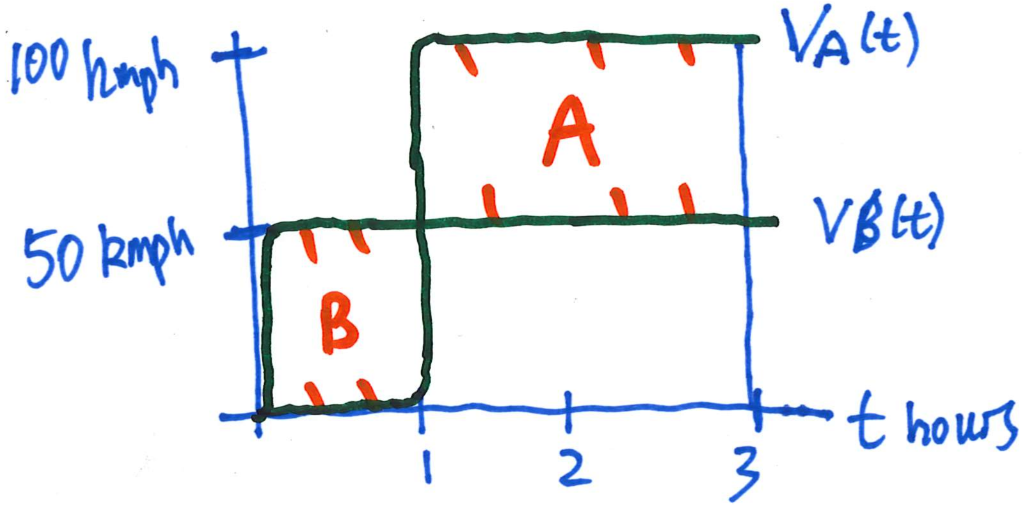
$$= 2\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) - 2(0+1)$$

$$= 4/\sqrt{2} - 2 = 2\sqrt{2} - 2.$$

$$\int_0^{\pi/2} (\cos x - \sin x)\,dx = A_1 - A_1 = 0$$

$$\int_0^{\pi/2} (\cos x - \sin x)\,dx = \text{Exercise}$$

Example 4

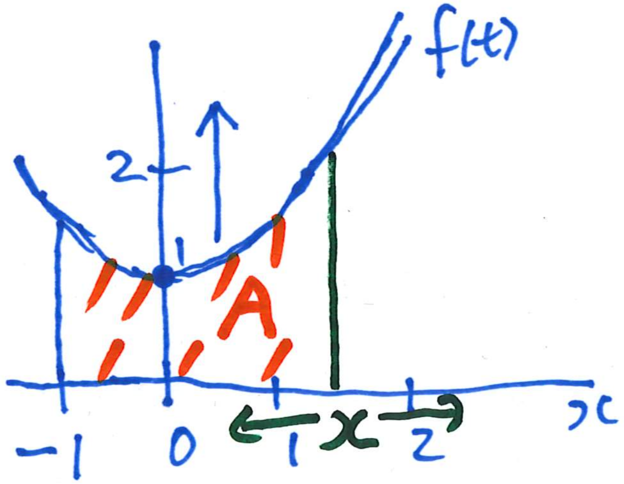


$$\int_0^3 (V_A(t) - V_B(t)) = \underbrace{A - B}_{>0} =$$

= the distance car A is ahead (if +ve) or behind (if −ve) car B after 3 hours.

Area Functions

Let $A(x) = \int_{-1}^{x} (1+t^2)\,dt$



FTC (2) $= \left[t + \frac{1}{3}t^3\right]_{-1}^{x} = \left(x + \frac{1}{3}x^3\right) - \left(-1 - \frac{1}{3}\right)$

$= x + \frac{1}{3}x^3 + 4/3$

$$\boxed{A(-1) = \int_{-1}^{-1} (1+t^2)\,dt = 0}$$

FTC (1) If $A(x) = \int_{-1}^{x} \underbrace{(1+t^2)}_{f(t)}\,dt$ then $A'(x) = f(x)$

$A'(x) = 1 + x^2$

$A(x) = \int (1+x^2)\,dx = x + \frac{1}{3}x^3 + \underline{C}$

$A(-1) = -1 - \frac{1}{3} + C = 0 \Rightarrow C = 4/3$

$A(x) = x + \frac{1}{3}x^3 + 4/3.$

Properties of Definite Integrals

We used $\displaystyle\int_a^b f(x)\,dx - \int_a^b g(x)\,dx = \int_a^b \big(f(x)-g(x)\big)\,dx$

Proof Let $F(x)$ and $\mathcal{E}(x)$ be antiderivatives of $f(x)$ and $g(x)$.
($\Leftrightarrow$ $F'(x)=f(x)$ and $\mathcal{E}'(x)=g(x)$)

$$\int_a^b f(x)\,dx - \int_a^b g(x)\,dx \overset{FTC(2)}{=} \Big[F(x)\Big]_a^b - \Big[\mathcal{E}(x)\Big]_a^b = \big(F(b)-F(a)\big)-\big(\mathcal{E}(b)-\mathcal{E}(a)\big)$$
$$= F(b)-F(a)-\mathcal{E}(b)+\mathcal{E}(a).$$

$$\int_a^b \big(f(x)-g(x)\big)\,dx = \Big[F(x)-\mathcal{E}(x)\Big]_a^b = \big(F(b)-\mathcal{E}(b)\big)-\big(F(a)-\mathcal{E}(a)\big) \ /\!/ ?$$
$$= F(b)-F(a)-\mathcal{E}(b)+\mathcal{E}(a)$$

① $\displaystyle\int_a^b c\,f(x)\,dx = c\int_a^b f(x)\,dx.$ ② $\displaystyle\int_a^b \big(f(x)\pm g(x)\big)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

③ $\displaystyle\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$ ④ $\displaystyle\int_a^a f(x)\,dx = 0$

⑤ $\displaystyle\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$