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Model Solutions
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Assignment #2

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MATH 800 Assignment #2

2.2.1. Rewrite each polynomial, ordering in (i) lex, (ii) grlex, (iii) grevlex order, giving LM(f), LT(f) and multideg(f).

a) $f(x,y,z) = 2x + 3y + z + x^2 - z^2 + x^3$

i) $f(x,y,z) = x^3 + x^2 + 2x + 3y - z^2 + z$, $LM(f) = LT(f) = x^3$, $\text{multideg}(f) = (3,0,0)$

ii) $f(x,y,z) = x^3 + x^2 - z^2 + 2x + 3y + z$, $LM(f) = LT(f) = x^3$, $\text{multideg}(f) = (3,0,0)$

iii) $f(x,y,z) = x^3 + x^2 - z^2 + 2x + 3y + z$, $LM(f) = LT(f) = x^3$, $\text{multideg}(f) = (3,0,0)$

b) $f(x,y,z) = 2x^2y^8 - 3x^5yz^4 + xyz^3 - xy^4$

i) $f(x,y,z) = -3x^5yz^4 + 2x^2y^8 - xy^4 + xyz^3$, $LM(f) = x^5yz^4$, $LT(f) = -3x^5yz^4$, $\text{multideg}(f) = (5,1,4)$

ii) $f(x,y,z) = -3x^5yz^4 + 2x^2y^8 - xy^4 + xyz^3$, $LM(f) = x^5yz^4$, $LT(f) = -3x^5yz^4$, $\text{multideg}(f) = (5,1,4)$

iii) $f(x,y,z) = 2x^2y^8 - 3x^5yz^4 - xy^4 + xyz^3$, $LM(f) = x^2y^8$, $LT = 2x^2y^8$, $\text{multideg}(f) = (2,8,0)$

2.2.2. Determine which monomial order was used in each of the following polynomials

a) $f(x,y,z) = 7x^2y^4z - 2xy^6 + x^2y^2$

grlex order (note: $xy^6 \not>_{lex} x^2y^2$, $x^2y^4z \not>_{grevlex} xy^6$)

b) $f(x,y,z) = xy^3z + xy^2z^2 + x^2z^3$

grevlex order (note: $xy^2z^2 \not>_{lex} x^2z^3$, $xy^2z^2 \not>_{grlex} x^2z^3$)

c) $f(x,y,z) = x^4y^5z + 2x^3y^2z - 4xy^2z^4$

lex order (note: $x^3y^2z \not>_{grlex} xy^2z^4$, $x^3y^2z \not>_{grevlex} xy^2z^4$)

2.2.4. Prove: grlex order is a monomial ordering on $k[x_1, \dots, x_n]$.

Proof: (i) Let $\alpha, \beta \in \mathbb{Z}^n_{\geq 0}$, and recall that $|\alpha| = \sum_{i=1}^n \alpha_i$, $|\beta| = \sum_{i=1}^n \beta_i$.

Case I: $|\alpha| \neq |\beta|$.

Then $|\alpha| > |\beta|$, so that $\alpha >_{grlex} \beta$ (by defn. of grlex)

or $|\alpha| < |\beta|$, so that $\beta >_{grlex} \alpha$ (by defn. of grlex).

Case II: $|\alpha| = |\beta|$.

Then $\alpha >_{grlex} \beta$ iff $\alpha >_{lex} \beta$ (by defn. of grlex).

Since $>_{lex}$ is a total ordering, it follows that

$\alpha >_{grlex} \beta$ or $\beta >_{grlex} \alpha$ or $\alpha =_{grlex} \beta$ (when $\alpha = \beta$).

$\therefore >_{grlex}$ is a total ordering on $\mathbb{Z}^n_{\geq 0}$. OK

(ii) Let $\alpha, \beta, \gamma \in \mathbb{Z}^n_{\geq 0}$, and suppose that $\alpha >_{grlex} \beta$.

Case I: $|\alpha| \neq |\beta|$.

Then $|\alpha| > |\beta|$

$$\Rightarrow |\alpha + \gamma| = \sum_{i=1}^n (\alpha_i + \gamma_i) = \sum_{i=1}^n \alpha_i + \sum_{i=1}^n \gamma_i = |\alpha| + |\gamma|$$

$$> |\beta| + |\gamma| = \sum_{i=1}^n \beta_i + \sum_{i=1}^n \gamma_i = \sum_{i=1}^n (\beta_i + \gamma_i) = |\beta + \gamma| \quad \text{OK}$$

$$\Rightarrow \alpha + \gamma >_{grlex} \beta + \gamma.$$

Case II: $|\alpha| = |\beta|$.

Then $|\alpha + \gamma| = |\alpha| + |\gamma| = |\beta| + |\gamma| = |\beta + \gamma|$.

So $\alpha + \gamma >_{grlex} \beta + \gamma$ iff $\alpha + \gamma >_{lex} \beta + \gamma$.

Now $\alpha >_{grlex} \beta \Rightarrow \alpha >_{lex} \beta$

$\Rightarrow \alpha + \gamma >_{lex} \beta + \gamma$ (since $>_{lex}$ is a monomial ordering) (property (ii)) ✓

$\therefore \alpha + \gamma >_{grlex} \beta + \gamma$.

(iii) Let $S \subset \mathbb{Z}^n_{\geq 0}$ be a nonempty subset.
We can partition S into subsets $S_i = \{\alpha \in S : |\alpha| = i\}$, $i \in \mathbb{Z}_{\geq 0}$. ✓
Let $A = \{i : S_i \neq \emptyset\} \subset \mathbb{Z}_{\geq 0}$. (A represents the possible values of $|\alpha|$ from S.) Note that $A \neq \emptyset$ since $S \neq \emptyset$.
Since $\mathbb{Z}_{\geq 0}$ is well-ordered, A has a least element i_0. ✓

Consider the nonempty subset $S_{i_0} \subset \mathbb{Z}^n_{\geq 0}$. For any $\alpha, \beta \in S_{i_0}$, $\alpha >_{grlex} \beta$ iff $\alpha >_{lex} \beta$ ✓ (by defn. of grlex), since $|\alpha| = |\beta| = i_0$.
Since $>_{lex}$ is a well-ordering on $\mathbb{Z}^n_{\geq 0}$, it follows that S_{i_0} has a least element α_0 under $>_{lex}$. ✓ BAD Subscript! × ~~[illegible] than these.~~
But then α_0 is the least element of S_i under $>_{grlex}$.
(Alternatively, there are $\binom{n+i_0-1}{i_0}$ elements $\alpha \in \mathbb{Z}^n_{\geq 0}$ with $|\alpha| = i_0$, so S_{i_0} must be finite and therefore has a least element α_0.) ✓

We claim that α_0 is the least element of S under $>_{grlex}$. ✓
Let $\beta \in S$. Then $|\beta| \geq |\alpha_0| = i_0$, for $|\beta| < i_0$ would contradict the minimality of i_0 in A. ✓
If $|\beta| > |\alpha_0|$ then $\beta >_{grlex} \alpha_0$ (by defn. of grlex). ✓
If $|\beta| = |\alpha_0|$ then $\beta \geq_{grlex} \alpha_0$ by the minimality of α_0 in S_{i_0} (since $\beta \in S_{i_0}$). ✓
$\therefore \beta \geq_{grlex} \alpha_0 \;\; \forall \beta \in S$. ✓

That is, S has a least element under $>_{grlex}$.
$\therefore >_{grlex}$ is a well-ordering on $\mathbb{Z}^n_{\geq 0}$. $\square$ ✓

2.3.1. Compute the remainder on division of $f = x^7y^2 + x^3y^2 - y + 1$ by the order set F using (i) grlex order, (ii) lex order.

a) $F = (xy^2 - x,\ x - y^3)$

i)

$q_1:\ x^6 + x^2$

$q_2:\ 0$

r

$$
\begin{array}{l|l}
xy^2 - x & x^7y^2 + x^3y^2 - y + 1 \\
-y^3 + x & -(x^7y^2 - x^7) \\
\hline
 & x^7 + x^3y^2 - y + 1 \\
 & x^3y^2 - y + 1 \longrightarrow x^7 \\
 & -(x^3y^2 - x^3) \\
\hline
 & x^3 - y + 1 \\
 & -y + 1 \longrightarrow x^7 + x^3 \\
 & 1 \longrightarrow x^7 + x^3 - y \\
 & 0 \longrightarrow x^7 + x^3 - y + 1
\end{array}
$$

$$\therefore f = (x^6 + x^2)(xy^2 - x) + 0(-y^3 + x) + \boxed{x^7 + x^3 - y + 1}$$ ✓

ii)

$q_1:\ x^6 + x^5y + x^4y^2 + x^4 + x^3y + x^2y^2 + 2x^2 + 2xy$

$q_2:\ x^6 + x^5y + x^4 + x^3y + 2x^2 + 2xy$

$$
\begin{array}{l|l}
xy^2 - x & x^7y^2 + x^3y^2 - y + 1 \\
x - y^3 & -(x^7y^2 - x^7) \\
\hline
 & x^7 + x^3y^2 - y + 1 \\
 & -(x^7 - x^6y^3) \\
\hline
 & x^6y^3 + x^3y^2 - y + 1 \\
 & -(x^6y^3 - x^6y) \\
\hline
 & x^6y + x^3y^2 - y + 1 \\
 & -(x^6y - x^5y^4) \\
\hline
 & x^5y^4 + x^3y^2 - y + 1 \\
 & -(x^5y^4 - x^5y^2) \\
\hline
 & x^5y^2 + x^3y^2 - y + 1 \\
 & -(x^5y^2 - x^5) \\
\hline
 & x^5 + x^3y^2 - y + 1 \\
 & -(x^5 - x^4y^3) \\
\hline
 & x^4y^3 + x^3y^2 - y + 1
\end{array}
$$

$$
\begin{array}{l}
x^4y^3 + x^3y^2 - y + 1 \\
-(x^4y^3 - x^4y) \\
\hline
x^4y + x^3y^2 - y + 1 \\
-(x^4y - x^3y^4) \\
\hline
x^3y^4 + x^3y^2 - y + 1 \\
-(x^3y^4 - x^3y^2) \\
\hline
2x^3y^2 - y + 1 \\
-(2x^3y^2 - 2x^3) \\
\hline
2x^3 - y + 1 \\
-(2x^3 - 2x^2y^3) \\
\hline
2x^2y^3 - y + 1 \\
-(2x^2y^3 - 2x^2y) \\
\hline
2x^2y - y + 1 \\
-(2x^2y - 2xy^4) \\
\hline
2xy^4 - y + 1
\end{array}
$$

(cont'd)

$a_1: +2y^2+2$

$a_2: +2$ $\qquad\qquad r$

$$
\begin{array}{rl}
 & 2xy^4 - y + 1 \\
 & -(2xy^4 - 2xy^2) \\
\hline
 & 2xy^2 - y + 1 \\
 & -(2xy^2 - 2x) \\
\hline
 & 2x - y + 1 \\
 & -(2x - 2y^3) \\
\hline
 & 2y^3 - y + 1 \\
 & -y + 1 \longrightarrow 2y^3 \\
 & 1 \longrightarrow 2y^3 - y \\
 & 0 \longrightarrow 2y^3 - y + 1
\end{array}
$$

$$\therefore f = (x^6+x^5y+x^4y^2+x^4+x^3y+x^2y^2+2x^2+2xy+2y^2+2)(xy^2-x) + (x^6+x^5y+x^4+x^3y+2x^2+2xy+2)(x-y^3) + \boxed{2y^3-y+1}$$

b) $F = (x-y^3,\ xy^2-x)$

gr-lex

i) $a_1: 0$

$a_2: x^6 + x^2$ $\qquad\qquad r$

$$
\begin{array}{r|l}
-y^3+x & x^7y^2 + x^3y^2 - y + 1 \\
xy^2-x & -(x^7y^2 - x^7) \\
\hline
 & x^7 + x^3y^2 - y + 1 \\
 & x^3y^2 - y + 1 \longrightarrow x^7 \\
 & -(x^3y^2 - x^3) \\
\hline
 & x^3 - y + 1 \\
 & -y + 1 \longrightarrow x^7 + x^3 \\
 & 1 \longrightarrow x^7 + x^3 - y \\
 & 0 \longrightarrow x^7 + x^3 - y + 1
\end{array}
$$

$$\therefore f = 0(-y^3+x) + (x^6+x^2)(xy^2-x) + \boxed{x^7+x^3-y+1}$$

(ii) a_1: $x^6y^2 + x^5y^5 + x^4y^8 + x^3y^{11} + x^2y^{14} + x^2y^2 + xy^{17} + xy^5 + y^{20} + y^8$

a_2:

$$
\begin{array}{l|l}
x - y^3 & \cancel{x^7y^2} + x^3y^2 - y + 1 \\
xy^2 - x & -(\cancel{x^7y^2} - x^6y^5) \\
\hline
 & \cancel{x^6y^5} + x^3y^2 - y + 1 \\
 & -(\cancel{x^6y^5} - x^5y^8) \\
\hline
 & \cancel{x^5y^8} + x^3y^2 - y + 1 \\
 & -(\cancel{x^5y^8} - x^4y^{11}) \\
\hline
 & \cancel{x^4y^{11}} + x^3y^2 - y + 1 \\
 & -(\cancel{x^4y^{11}} - x^3y^{14}) \\
\hline
 & \cancel{x^3y^{14}} + x^3y^2 - y + 1 \\
 & -(\cancel{x^3y^{14}} - x^2y^{17}) \\
\hline
 & \cancel{x^3y^2} + x^2y^{17} - y + 1 \\
 & -(\cancel{x^3y^2} - x^2y^5) \\
\hline
 & \cancel{x^2y^{17}} + x^2y^5 - y + 1 \\
 & -(\cancel{x^2y^{17}} - xy^{20}) \\
\hline
 & \cancel{x^2y^5} + xy^{20} - y + 1 \\
 & -(\cancel{x^2y^5} - xy^8) \\
\hline
 & \cancel{xy^{20}} + xy^8 - y + 1 \\
 & -(\cancel{xy^{20}} - y^{23}) \\
\hline
 & \cancel{xy^8} + y^{23} - y + 1 \\
 & -(\cancel{xy^8} - y^{11}) \\
\hline
 & y^{23} + y^{11} - y + 1
\end{array}
\qquad r
$$

$$
\begin{aligned}
y^{11} - y + 1 &\longrightarrow y^{23} \\
-y + 1 &\longrightarrow y^{23} + y^{11} \\
1 &\longrightarrow y^{23} + y^{11} - y \\
0 &\longrightarrow y^{23} + y^{11} - y + 1
\end{aligned}
$$

$$
\therefore f = (x^6y^2 + x^5y^5 + x^4y^8 + x^3y^{11} + x^2y^{14} + x^2y^2 + xy^{17} + xy^5 + y^{20} + y^8)(x - y^3) + 0(xy^2 - x) + \boxed{y^{23} + y^{11} - y + 1}
$$

2.3.3. [See Maple worksheets]

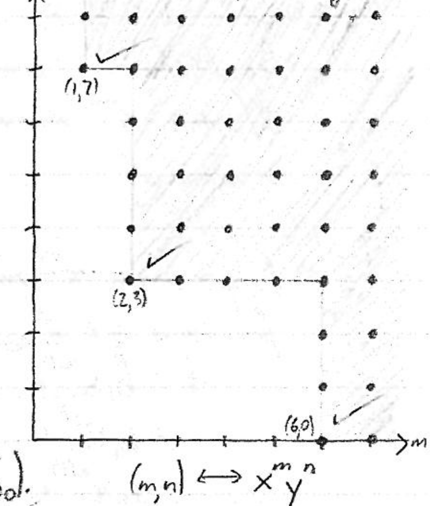
2.3.5. [See Maple worksheets]

2.4.3. Let $I = \langle x^6, x^2y^3, xy^7 \rangle \subset k[x,y]$.

a) Plot the set of exponent vectors of monomials $x^m y^n$ appearing in elements of I.

Any $f \in I$ is a k-linear combination of the monomials in I (Lemma 3), and any monomial in I is divisible by x^6, x^2y^3 or xy^7 (Lemma 2).

$\therefore$ the set of exponent vectors of monomials appearing in elements of I is
$$((6,0)+\mathbb{Z}^2_{\geq 0}) \cup ((2,3)+\mathbb{Z}^2_{\geq 0}) \cup ((1,7)+\mathbb{Z}^2_{\geq 0}).$$



$(m,n) \longleftrightarrow x^m y^n$

b) Applying the division algorithm to $f \in k[x,y]$ with divisors $\{x^6, x^2y^3, xy^7\}$, what terms can appear in the remainder?

By the division algorithm,
$$f = a_1x^6 + a_2x^2y^3 + a_3xy^7 + r$$
where $a_1, a_2, a_3 \in k[x,y]$ and no term in the remainder r is divisible by $LT(x^6) = x^6$, $LT(x^2y^3) = x^2y^3$ or $LT(xy^7) = xy^7$.

Thus the terms in r can only contain monomials not in I, that is, terms of the form

cy^n, $n \geq 0$ or

cxy^n, $0 \leq n \leq 6$ or

cx^my^n, $2 \leq m \leq 5$, $0 \leq n \leq 2$

($c \in k$).

2.4.8. a) Prove that every monomial ideal has a minimal basis.

Proof: Let $I \subset k[x_1, \dots, x_n]$ be a monomial ideal.
By Dickson's Lemma I has a finite basis $\{x^{\alpha(1)}, \dots, x^{\alpha(s)}\}$.
If no $x^{\alpha(i)}$ divides another $x^{\alpha(j)}$, $i \neq j$, then this basis is minimal. ✓

Suppose ~~an $x^{\alpha(i)}$ does divide another $x^{\alpha(j)}$~~ $x^{\alpha(i)} \mid x^{\alpha(j)}$ for some $i \neq j$, so $x^{\alpha(j)} = x^{\gamma} x^{\alpha(i)}$ ✓
for some $\gamma \in \mathbb{Z}^n_{\geq 0}$. Take any $f \in I$.

$$\begin{aligned} f &= h_1 x^{\alpha(1)} + \dots + h_i x^{\alpha(i)} + \dots + h_j x^{\alpha(j)} + \dots + h_s x^{\alpha(s)} \quad \text{for some } h_i \in k[x_1, \dots, x_n] \\ &= h_1 x^{\alpha(1)} + \dots + h_i x^{\alpha(i)} + \dots + h_j x^{\gamma} x^{\alpha(i)} + \dots + h_s x^{\alpha(s)} \\ &= h_1 x^{\alpha(1)} + \dots + (h_i + h_j x^{\gamma}) x^{\alpha(i)} + \dots + h_s x^{\alpha(s)} \\ &\in \langle x^{\alpha(k)} : k \in \{1, \dots, s\} \setminus \{j\} \rangle \subset I. \end{aligned}$$

So $\langle x^{\alpha(k)} : k \in \{1, \dots, s\} \setminus \{j\} \rangle = I$, and $\{x^{\alpha(1)}, \dots, x^{\alpha(s)}\} \setminus \{x^{\alpha(j)}\}$ is a smaller basis for I. ✓
If this new basis is not minimal then we can repeat with another $x^{\alpha(i)}$ that divides another $x^{\alpha(j)}$. ✓

Each time we repeat get a new basis with one less monomial as the previous. Since we started with a finite basis of s monomials, after at most $s-1$ repetitions we must get a basis where no $x^{\alpha(i)}$ divides another $x^{\alpha(j)}$, or where there is only one monomial $x^{\alpha(i)}$.
In either case, we have a minimal basis for I. ✓ $\square$

b) Show that every monomial ideal has a unique minimal basis.

Proof: Let $A = \{x^{\alpha(1)}, \dots, x^{\alpha(s)}\}$ and $B = \{x^{\beta(1)}, \dots, x^{\beta(t)}\}$ be minimal bases for a monomial ideal $I \subset k[x_1, \dots, x_n]$.
Consider $x^{\beta(j)} \in B \subset I$. $x^{\beta(j)}$ is divisible by some $x^{\alpha(i)} \in A$ since A is a basis for I (Lemma 2). So $x^{\beta(j)} = x^{\gamma} x^{\alpha(i)}$.
But $x^{\alpha(i)}$ is divisible by some $x^{\beta(k)} \in B$ since B is a basis for I (Lemma 2). So $x^{\alpha(i)} = x^{\delta} x^{\beta(k)}$. ✓
Then $x^{\beta(j)} = x^{\gamma} x^{\delta} x^{\beta(k)}$, so $x^{\beta(k)}$ divides $x^{\beta(j)}$.

However B is a minimal basis, so we must have $x^{\beta(k)} = x^{\beta(j)}$ ✓
$(k=j)$ and $x^{\gamma} = x^{\delta} = 1$. ✓
Then $x^{\beta(j)} = x^{\alpha(i)} \in A$.
So $B \subset A$, and conversely $A \subset B$. OK
$\therefore A = B$ and the minimal basis is unique. $\square$ ✓ NICE

2.4.10. Given $k[x_1, \dots, x_n, y_1, \dots, y_m]$, define a monomial order $>_{mixed}$ by
$x^{\alpha}y^{\beta} >_{mixed} x^{\gamma}y^{\delta} \iff x^{\alpha} >_{lex} x^{\gamma}$, or $x^{\alpha} = x^{\gamma}$ and $y^{\beta} >_{grlex} y^{\delta}$
$(\alpha, \gamma \in \mathbb{Z}^n_{\geq 0},\ \beta, \delta \in \mathbb{Z}^m_{\geq 0})$.
Prove that $>_{mixed}$ is a monomial order (a product order).

Proof: As notation we will let $(\alpha, \beta) \in \mathbb{Z}^n_{\geq 0} \times \mathbb{Z}^m_{\geq 0}$ be the exponent vector for $x^{\alpha}y^{\beta} \in k[x_1, \dots, x_n, y_1, \dots, y_m]$. ✓

(i) Let $(\alpha, \beta), (\gamma, \delta) \in \mathbb{Z}^n_{\geq 0} \times \mathbb{Z}^m_{\geq 0}$.
Since $>_{lex}$ is a total ordering on $\mathbb{Z}^n_{\geq 0}$, either
$\alpha >_{lex} \gamma$, so $(\alpha, \beta) >_{mixed} (\gamma, \delta)$ (by defn of $>_{mixed}$), ✓
OR $\gamma >_{lex} \alpha$, so $(\gamma, \delta) >_{mixed} (\alpha, \beta)$ (by defn of $>_{mixed}$), ✓
OR $\alpha = \gamma$. ✓
Suppose we are in this third case ($\alpha = \gamma$).
Then since $>_{grlex}$ is a total ordering on $\mathbb{Z}^m_{\geq 0}$, either
$\beta >_{grlex} \delta$, so $(\alpha, \beta) >_{mixed} (\gamma, \delta)$ (by defn of $>_{mixed}$), ✓
OR $\delta >_{grlex} \beta$, so $(\gamma, \delta) >_{mixed} (\alpha, \beta)$ (by defn of $>_{mixed}$), ✓
OR $\beta = \delta$, so $(\alpha, \beta) = (\gamma, \delta)$. ✓
$\therefore >_{mixed}$ is a total ordering on $\mathbb{Z}^n_{\geq 0} \times \mathbb{Z}^m_{\geq 0}$. ✓

ii) Let $(\alpha, \beta), (\gamma, \delta), (\rho, \sigma) \in \mathbb{Z}^n_{\geq 0} \times \mathbb{Z}^m_{\geq 0}$, and suppose that $(\alpha, \beta) >_{mixed} (\gamma, \delta)$. ✓

Case I: $\alpha >_{lex} \gamma$. Then $\alpha + \rho >_{lex} \gamma + \rho$, since $>_{lex}$ is a monomial ordering (property (ii)). ✓
So $(\alpha, \beta) + (\rho, \sigma) = (\alpha + \rho, \beta + \sigma)$
$>_{mixed} (\gamma + \rho, \delta + \sigma)$
$= (\gamma, \delta) + (\rho, \sigma)$. ✓

Case II: $\alpha = \gamma$ and $\beta >_{grlex} \delta$. Then $\beta+\sigma >_{grlex} \delta+\sigma$ since $>_{grlex}$ is a monomial ordering (property (ii)). ✓

Also $\alpha+\rho = \gamma+\rho$.

So $(\alpha,\beta)+(\rho,\sigma) = (\alpha+\rho, \beta+\sigma)$ ✓

$>_{mixed} (\gamma+\rho, \delta+\sigma)$ ✓

$= (\gamma,\delta)+(\rho,\sigma)$. ✓

NICE

iii) Let $(\alpha,\beta) \in \mathbb{Z}^n_{\geq 0} \times \mathbb{Z}^m_{\geq 0}$.

Since $>_{lex}$ is a monomial ordering, it is a well-ordering on $\mathbb{Z}^n_{\geq 0}$.

So by Corollary 6, $\alpha \geq_{lex} 0 \in \mathbb{Z}^n_{\geq 0}$. ✓

If $\alpha >_{lex} 0$ then $(\alpha,\beta) >_{mixed} (0,0) \in \mathbb{Z}^n_{\geq 0} \times \mathbb{Z}^m_{\geq 0}$ (by defn of $>_{mixed}$).

Suppose $\alpha =_{lex} 0$. Then $\alpha = 0$. ✓

Since $>_{grlex}$ is a monomial ordering, it is a well-ordering on $\mathbb{Z}^m_{\geq 0}$.

So by Corollary 6, $\beta \geq_{grlex} 0 \in \mathbb{Z}^m_{\geq 0}$. ✓

If $\beta >_{grlex} 0$ then $(\alpha,\beta) >_{mixed} (0,0) \in \mathbb{Z}^n_{\geq 0} \times \mathbb{Z}^m_{\geq 0}$ (by defn of $>_{mixed}$).

If $\beta =_{grlex} 0$ then $\beta = 0$, so $(\alpha,\beta) = (0,0)$. Thus $(\alpha,\beta) =_{mixed} (0,0)$. ✓

$\therefore$ $(\alpha,\beta) \geq_{mixed} (0,0) \in \mathbb{Z}^n_{\geq 0} \times \mathbb{Z}^m_{\geq 0}$. So by Corollary 6, $>_{mixed}$ is a well-ordering of $\mathbb{Z}^n_{\geq 0} \times \mathbb{Z}^m_{\geq 0}$. ✓

$\therefore$ $>_{mixed}$ is a monomial ordering on $k[x_1,\dots,x_n,y_1,\dots,y_m]$. ✓ □

2.5.1. Let $I = \langle g_1, g_2, g_3 \rangle \subset \mathbb{R}[x,y,z]$, $g_1 = xy^2 - xz + y$, $g_2 = xy - z^2$, $g_3 = x - yz^4$.
Find a $g \in I$ s.t. $LT(g) \notin \langle LT(g_1), LT(g_2), LT(g_3) \rangle$. (Use lex order.)

Let $g = g_2 - yg_3 = (xy - z^2) - (xy - y^2z^4) = y^2z^4 - z^2 \in I$.
Then $LT(g) = y^2z^4 \notin \langle LT(g_1), LT(g_2), LT(g_3) \rangle = \langle xy^2, xy, x \rangle$, ✓ by Lemma 2 from §4, because this is a monomial ideal and y^2z^4 is not divisible by any of xy^2, xy or x. ✓

$\therefore$ $\{g_1, g_2, g_3\}$ is not a Groebner basis for I w.r.t. lex order. ✓

2.5.6. Give a different proof from Corollary 6: Let $I \subset k[x_1,\dots,x_n]$ be an ideal other than $\{0\}$. If $G = \{g_1,\dots,g_s\} \subset I$ satisfies
$\langle LT(I)\rangle = \langle LT(g_1),\dots, LT(g_s)\rangle$ (ie. if G is Groebner basis for I)
Then $I = \langle g_1,\dots, g_s\rangle$ (G is a basis for I).

Proof: Clearly $\langle g_1,\dots,g_s\rangle \subset I$ since $G \subset I$. We must show that $I \subset \langle g_1,\dots,g_s\rangle$.

Let $f \in I$. Consider dividing f by G. We will show that in the division algorithm presented in class and on this assignment, each time we reach the condition "while $p \neq 0$" beginning the loop, the following two loop invariants hold:
① $p \in I$, and ② $r = 0$. ✓

- The first time we check this condition, p and r have been assigned the values $p = f \in I$ and $r = 0$. ✓
- For each subsequent time we check the condition $p \neq 0$ we have just completed a previous iteration of the loop. Let p_{prev} and r_{prev} denote the values of p and r at the beginning of this previous iteration, where $p_{prev} \in I$ and $r_{prev} = 0$. ✓

$\Rightarrow LT(p_{prev}) \in \langle LT(I)\rangle = \langle LT(g_1),\dots, LT(g_s)\rangle$.
This is a monomial ideal, so some $LT(g_i)$ divides $LT(p)$. ✓
Then the algorithm computed $p = p_{prev} - t \cdot g_i \in I$ ($t \in k[x_1,\dots,x_n]$) ✓
and $r = r_{prev} = 0$ was unchanged. ✓ ~~I know what you need~~
So when we reach the condition the loop invariants hold. ~~but you need to write down~~

Therefore when the condition finally fails and the algorithm terminates, we still have $p \in I$ and $r = 0$. ✓
This shows that $f = a_1 g_1 + \dots + a_s g_s + 0 \in \langle g_1,\dots, g_s\rangle$.
So $I \subset \langle g_1,\dots, g_s\rangle$.

✓
$\therefore I = \langle g_1,\dots, g_s\rangle$. (A Groebner basis is a basis.) $\square$

2.5.7. Is $\{f_1 = x^4y^2 - z^5,\ f_2 = x^3y^3 - 1,\ f_3 = x^2y^4 - 2z\}$ a Groebner basis for $I = \langle f_1, f_2, f_3\rangle$ w.r.t. grlex order, $x > y > z$? Why or why not?

Let $f = yf_2 - xf_3 = (x^3y^4 - y) - (x^3y^4 - 2xz) = 2xz - y$.
Then $f \in I = \langle f_1, f_2, f_3\rangle$.

TAC Suppose $LT(f) = 2xz \in \langle LT(f_1), LT(f_2), LT(f_3)\rangle = \langle x^4y^2, x^3y^3, x^2y^4\rangle$.
Then $\frac{1}{2}LT(f) = xz \in \langle x^4y^2, x^3y^3, x^2y^4\rangle$.
Since xz is a monomial and $\langle x^4y^2, x^3y^3, x^2y^4\rangle$ is a monomial ideal,
Too wordy here. This implies xz is divisible by one of x^4y^2, x^3y^3 or x^2y^4.
But xz is not divisible by any of these, giving a contradiction.
Therefore $LT(f) \notin \langle LT(f_1), LT(f_2), LT(f_3)\rangle$. But $LT(f) \in \langle LT(I)\rangle$,
so $\langle LT(I)\rangle \notin \langle LT(f_1), LT(f_2), LT(f_3)\rangle$.
$\therefore \{f_1, f_2, f_3\}$ is not a Groebner basis for $I = \langle f_1, f_2, f_3\rangle$ w.r.t. grlex order.

2.5.10. Let $I \subset k[x_1, \dots, x_n]$ be a principal ideal. Show that any finite subset of I containing a generator for I is a Groebner basis for I.

Proof: Since I is a principal ideal, $\exists$ a generator $f_1 \in k[x_1, \dots, x_n]$ s.t. $I = \langle f_1\rangle$.
Let $S \subset I$ be any finite subset containing f_1.
Clearly $\langle LT(g) : g \in S\rangle \subset \langle LT(I)\rangle = \langle LT(g) : g \in I\rangle$.

Let $f \in I$. Then $I = \langle f_1\rangle \Rightarrow f = af_1$ for some $a \in k[x_1, \dots, x_n]$.
Now w.r.t. whatever monomial order we are using,

$$a = c_\alpha x^\alpha + \sum_{\gamma < \alpha} c_\gamma x^\gamma \quad \text{and} \quad f_1 = d_\beta x^\beta + \sum_{\delta < \beta} d_\delta x^\delta,$$

where the c's and d's are in k, $\alpha = \text{multideg}(a)$ and $\beta = \text{multideg}(f_1)$.
Then

$$a \cdot f_1 = c_\alpha d_\beta x^\alpha x^\beta + \sum_{\delta < \beta} c_\alpha d_\delta x^\alpha x^\delta + \sum_{\substack{\gamma < \alpha \\ \delta \le \beta}} c_\gamma d_\delta x^\gamma x^\delta$$
$$= c_\alpha d_\beta x^{\alpha+\beta} + \sum_{\delta < \beta} c_\alpha d_\delta x^{\alpha+\delta} + \sum_{\substack{\gamma < \alpha \\ \delta \le \beta}} c_\gamma d_\delta x^{\gamma+\delta}.$$

and $LT(af_1) = c_\alpha d_\beta x^{\alpha+\beta}$ is okay.

Now in the first sum, $\beta > \delta \Rightarrow \alpha + \beta > \alpha + \delta$
and in the second sum, $\alpha > \gamma$ and $\beta \ge \delta \Rightarrow \alpha + \beta \ge \alpha + \delta > \gamma + \delta$
(by property (ii) of the monomial order).
So $\text{multideg}(af_1) = \alpha + \beta$, and

$$LT(f) = LT(af_1) = c_\alpha d_\beta x^{\alpha+\beta} = (c_\alpha x^\alpha)(d_\beta x^\beta) = LT(a) \cdot LT(f_1).$$

(This is basically the result of Lemma 8 from §2.)

NICE ☺

The point of all this is that, since $LT(f_1)$ divides $LT(f)$,
$LT(f) \in \langle LT(f_1)\rangle \subset \langle LT(g) : g \in S\rangle$.
So $\langle LT(I)\rangle \subset \langle LT(g) : g \in S\rangle$. ✓
Then $\langle LT(I)\rangle = \langle LT(g) : g \in S\rangle$. ✓
$\therefore$ S is a Groebner basis for I. □ ✓

2.5.12. Show that the Ascending Chain Condition implies the Hilbert Basis Theorem.

Proof: We are given the hypothesis that every ascending chain of ideals in $k[x_1, \dots, x_n]$ stabilizes.
TAC Assume $\exists$ an ideal $I \subset k[x_1, \dots, x_n]$ that has no finite basis (generating set). ✓
$I \neq \{0\}$, since $\{0\}$ is generated by $\{0\}$ (or by $\emptyset$). ✓
So choose $f_1 \in I$. Since $\{f_1\}$ is not a basis, $\langle f_1\rangle \neq I$ and $I \setminus \langle f_1\rangle \neq \emptyset$. ✓
So choose $f_2 \in I \setminus \langle f_1\rangle$. Since $\{f_1, f_2\}$ is not a basis, $\langle f_1, f_2\rangle \neq I$ and $I \setminus \langle f_1, f_2\rangle \neq \emptyset$.
So choose $f_3 \in I \setminus \langle f_1, f_2\rangle$. ✓
We can continue this process indefinitely:
since $\{f_1, \dots, f_i\} \overset{\subset I}{}$ is not a basis for I (because I has no finite basis),
$\langle f_1, \dots, f_i\rangle \neq I$ and we can choose $f_{i+1} \in I \setminus \langle f_1, \dots, f_i\rangle \neq \emptyset$.
Notice that $\forall i \in \mathbb{N}$, $f_{i+1} \notin \langle f_1, \dots, f_i\rangle$.
Thus we have an infinite ascending chain of ideals
$\langle f_1\rangle \subsetneq \langle f_1, f_2\rangle \subsetneq \langle f_1, f_2, f_3\rangle \subsetneq \dots$
that never stabilizes.
This contradicts our hypothesis (the ACC).
$\therefore$ every ideal $I \subset k[x_1, \dots, x_n]$ must have a finite basis. ✓ □

* 2.5.13. Let $V_1 \supset V_2 \supset V_3 \supset \dots$ be a descending chain of affine varieties.
Show that there is some $N \geq 1$ such that $V_N = V_{N+1} = V_{N+2} = \dots$.

Proof: By Proposition 8 (i) of Chapter 1, §4,
$V_1 \supset V_2 \supset V_3 \supset \dots \Rightarrow \mathbb{I}(V_1) \subset \mathbb{I}(V_2) \subset \mathbb{I}(V_3) \subset \dots$
This gives an ascending chain of ideals, so by the ACC $\exists$ some $N \geq 1$ s.t. $\mathbb{I}(V_N) = \mathbb{I}(V_{N+1}) = \mathbb{I}(V_{N+2})$. Then part (ii) of the same Proposition 8 gives $V_N = V_{N+1} = V_{N+2} = \dots$. □ ✓

2.5.14. Let $f_1, f_2, \ldots \in k[x_1, \ldots, x_n]$ be an infinite collection of polynomials and let $I = \langle f_1, f_2, \ldots \rangle$ be the ideal they generate. Prove that there is an integer N such that $I = \langle f_1, \ldots, f_N \rangle$.

Proof: Let $I_i = \langle f_1, \ldots, f_i \rangle \quad \forall i \in \mathbb{N}$.

Claim: $I = \bigcup_{i=1}^{\infty} I_i$.

Each $I_i = \langle f_1, \ldots, f_i \rangle \subset \langle f_1, f_2, \ldots \rangle = I$, so $\bigcup_{i=1}^{\infty} I_i \subset I$. ✓ ✓

Let $f \in I$. Recall that a polynomial is a finite sum of nonzero terms, so f must be a finite sum // Don't agree. Consider

I is an ideal $\Rightarrow f = \sum a_i f_i, \quad a_i \in k[x_1, \ldots, x_n] \setminus \{0\}$.

Then let $j = \max\{i \in \mathbb{N} : a_i f_i$ is in this sum for $f\}$.

(−1) So $f \in I_j = \langle f_1, \ldots, f_j \rangle \subset \bigcup_{i=1}^{\infty} I_i$.

Therefore $I \subset \bigcup_{i=1}^{\infty} I_i$, so $I = \bigcup_{i=1}^{\infty} I_i$.

$$\overset{f_1 \; f_2 \; f_3 \; \cdots}{I = \langle 2, -1, +3, -3, 4, -4, \ldots} \quad \Downarrow \quad f = 1 = (f_1 + f_2) + (f_3 + f_4) + f_5 + \cdots$$

Also, $I_1 \subset I_2 \subset I_3 \subset \ldots$ form an ascending chain of ideals.

By the ACC $\exists$ some $N \geq 1$ s.t. $I_N = I_{N+1} = I_{N+2} = \ldots$.

Then $\bigcup_{i=1}^{\infty} I_i = \bigcup_{i=1}^{N} I_i = I_N$. ✓

$\therefore I = I_N = \{f_1, \ldots, f_N\}$. □ ✓

2.6.2. [See Maple worksheets]

2.6.3. Show that if $G = \{g_1, \ldots, g_t\}$ is a basis for $I = \langle g_1, \ldots, g_t \rangle \subset k[x_1, \ldots, x_n]$ s.t. $\overline{f}^G = 0 \;\; \forall f \in I$, then G is a Groebner basis for I.

Proof: Suppose TAC that G is not a Groebner basis. /

Then $\exists f \in I$ s.t. $LT(f) \notin \langle LT(g_1), \ldots, LT(g_t) \rangle$ (a monomial ideal).

Then $LT(f)$ is not divisible by any of $LT(g_1), \ldots, LT(g_t)$. ✓

Therefore in the first iteration of the "while $p \neq 0$" loop in the division algorithm (as we presented), $LT(f)$ is transferred from p to r. ✓

ie. at the beginning of the algorithm $p = f$ and $r = 0$,

and in the first iteration we set $p := p - LT(p) = f - LT(f)$ and ✓

$r = LT(p) = LT(f)$.

An important observation now is that at every step of the division algorithm, we never introduce a term to p that is greater in the monomial ordering than or equal to $LT(p)$:

If at some step no $LT(g_i)$ divides $LT(p)$ then $LT(p)$ is transferred to r and no terms are added to p.

If instead $LT(g_i) | LT(p)$ then

$$p := p - \frac{LT(p)}{LT(g_i)} g_i \quad \text{where } LT\left(\frac{LT(p)}{LT(g_i)} g_i\right) = LT(p),$$

which cancels with the leading term of p, leaving only terms with smaller multidegree than $LT(p)$.

Then after the first iteration of the loop, any additional terms transferred from p to r will have multidegree less than or equal to

$\text{multideg}(f - LT(f)) < \text{multideg}(f)$.

So nothing can cancel out the $LT(f)$ originally transferred to r. ✓

Therefore when the algorithm terminates, the remainder $r \neq 0$. ✓

This contradicts $\overline{f}^G = 0$.

Therefore G must be a Groebner basis. □ ✓

2.6.9. [See Maple worksheets]

* 2.6.12 Let $I \subset k[x_1, \ldots, x_n]$ be an ideal, and let G be a Groebner basis of I.

a) Show that $\overline{f}^G = \overline{g}^G$ iff $f - g \in I$.

Proof: Let $f, g \in k[x_1, \ldots, x_n]$. Then by Proposition 1 there are unique $f', g' \in I$ and unique $r, r' \in k[x_1, \ldots, x_n]$ s.t.

$f = f' + r$ and $g = g' + r'$,

and no term of r, r' is divisible by any element of $\langle LT(I) \rangle$. ✓

(Note: because $\langle LT(I) \rangle = \langle LT(g) : g \in G \rangle$ is a monomial ideal.

Also uniqueness of f', g' follows from uniqueness of r, r') ✓

Now if $\overline{f}^G = \overline{g}^G$ then $r = r'$ since $r = \overline{f}^G$ and $r' = \overline{g}^G$. ✓

So $f - g = (f' + r) - (g' + r') = f' - g' \in I$ since $f', g' \in I$. ✓

Conversely if $f-g\in I$ then $(f-g)-f'+g' = r-r' \in I$. ✓
If $r-r' \neq 0$ then $LT(r-r') \in \langle LT(I)\rangle$ so $LT(r-r')$ is divisible by some element of $\langle LT(I)\rangle$. But this is impossible since no term of r, r' is divisible by any element of $\langle LT(I)\rangle$. Thus $r-r'$ must be 0 ✓

$\Rightarrow r = r'$ ✓

$\Rightarrow \overline{f}^G = \overline{g}^G$. $\square$

b) Deduce that $\overline{f+g}^G = \overline{f}^G + \overline{g}^G$.

Proof: Given $f = f'+r$, $g = g'+r'$ as before, we have

$f+g = (f'+g') + (r+r')$

where $f'+g' \in I$ and no term of $r+r'$ is divisible by any element of $\langle LT(I)\rangle$. Also, $f'+g'$ and $r+r'$ are the unique polynomials with these properties.

So $\overline{f+g}^G = r+r'$. But $r = \overline{f}^G$ and $r' = \overline{g}^G$.

$\therefore \overline{f+g}^G = \overline{f}^G + \overline{g}^G$. $\square$ ✓

c) Deduce that $\overline{fg}^G = \overline{\overline{f}^G \cdot \overline{g}^G}^G$.

Proof: Given $f = f'+r$, $g = g'+r'$ as before, we have

$f\cdot g = (f'+r)(g'+r') = f'g' + rg' + r'f' + rr'$ ✓

where $f'g', rg', r'f' \in I$. ✓

Now by Proposition 1 there are unique $h\in I$ and $r'' \in k[x_1,\dots,x_n]$ s.t. $rr' = h + r''$ and no term of r'' is divisible by any element of $\langle LT(I)\rangle$. ✓

Then $f\cdot g = \underbrace{(f'g' + rg' + r'f' + h)}_{\in I} + r''$ ← no term divisible by any element of $\langle LT(I)\rangle$, ✓

and these polynomials are unique with these properties.

So $\overline{f\cdot g}^G = r'' = \overline{r\cdot r'}^G = \overline{\overline{f}^G \cdot \overline{g}^G}^G$. $\square$ ✓

Additional Problem 1: [See Maple worksheets]

Additional Problem 2: Fix a monomial order $>$ on $\mathbb{Z}^n_{\geq 0}$ and let $f, f_1, \dots, f_s \in k[x_1, \dots, x_n]$. Suppose we are dividing f by $(f_1, \dots, f_s)$. Prove that if the division algorithm terminates then
$$f = a_1 f_1 + \dots + a_s f_s + r$$
for some r satisfying no term of r is divisible by any $LT(f_i)$, $1 \leq i \leq s$.

Proof: We will show that, in the division algorithm, each time we reach the condition "while $p \neq 0$" beginning the loop, the loop invariant
$$f - a_1 f_1 - \dots - a_s f_s - (p + r) = 0$$
holds.

The first time we reach "while $p \neq 0$" the algorithm has just assigned $p = f$, $r = 0$ and $a_i = 0$, $1 \leq i \leq s$, so
$$f - a_1 f_1 - \dots - a_s f_s - (p + r) = f - 0 - \dots - 0 - (f + 0) = 0. \checkmark$$

If $f - a_1 f_1 - \dots - a_s f_s - (p + r) = 0$ at ~~some time~~ step $k-1$ or the k-1st time round the loop ... when we reach "while $p \neq 0$", let us see what happens when we execute the next iteration of the loop.

- If $LT(p)$ is divisible by some $LT(f_i)$ then the algorithm computes new values $a_i' = a_i + \frac{LT(p)}{LT(f_i)}$ and $p' = p - \frac{LT(p)}{LT(f_i)} f_i$.

Then
$$f - a_1 f_1 - \dots - a_i' f_i - \dots - a_s f_s - (p' + r)$$
$$= f - a_1 f_1 - \dots - \left(a_i + \frac{LT(p)}{LT(f_i)}\right) f_i - \dots - a_s f_s - \left(\left(p - \frac{LT(p)}{LT(f_i)} f_i\right) + r\right)$$
$$= f - a_1 f_1 - \dots - a_i f_i - \cancel{\frac{LT(p)}{LT(f_i)} f_i} - \dots - a_s f_s - p + \cancel{\frac{LT(p)}{LT(f_i)} f_i} - r$$
$$= f - a_1 f_1 - \dots - a_s f_s - (p + r) = 0 \checkmark$$
the next time we reach "while $p \neq 0$".

- If $LT(p)$ is divisible by none of the $LT(f_i)$ then the algorithm computes new values $r' = r + LT(p)$ and $p' = p - LT(p)$.

Then

$$f - a_1 f_1 - \dots - a_s f_s - (p' + r')$$
$$= f - a_1 f_1 - \dots - a_s f_s - ((p - LT(p)) + (r + LT(p)))$$
$$= f - a_1 f_1 - \dots - a_s f_s - (p + r) = 0$$
the next time we reach "while $p \neq 0$".

Thus the loop invariant holds each time we reach "while $p \neq 0$". If the division algorithm terminates then $p = 0$, ~~some time when we reach this condition~~. Then
$$f - a_1 f_1 - \dots - a_s f_s - (p + r) = 0$$
$$\Rightarrow f - a_1 f_1 - \dots - a_s f_s - (0 + r) = 0$$
$$\Rightarrow f = a_1 f_1 + \dots + a_s f_s + r \quad \checkmark$$
if the algorithm terminates.

Notice that $LT(p)$ is added to r only if $LT(p)$ is not divisible by any of the $LT(f_i)$. So we get the desired property that no term of r is divisible by any $LT(f_i)$. $\square$ $\checkmark$

MATH 800 Assignment 2
(due June 7, 2006, 9:30)

```
> restart;
> X := [x,y,z]:
```

2.3.3

Implementation of the Division Algorithm

We program the multivariate division algorithm. Our procedure takes as input a polynomial f to divide, an ordered s-tuple $[f(1), ..., f(s)]$ to divide by, a variable ordering X, and a procedure $LT(g, X)$ that computes the leading term of a polynomial g with respect to a certain monomial order. The output from our procedure is $a_1, ..., a_s, r$ satisfying the conditions given in the division algorithm. The procedure uses a global procedure monom_divis that tests whether one monomial term is divisible by another monomial term. If the optional parameter *verbose = true* is given, the procedure will print out its intermediate calculations.

```
> DIVIDE := proc( f, fL::list, X::list, LT::procedure )
  local s, a, r, p, t, i, verb;
  global monom_divis;        NOT NECESSARY BUT OKAY.
    verb := false;
    if (nargs > 4) then for t in args[5..-1] do
      if (op(1,t) = verbose) then verb := op(2,t) fi;
    od fi;
    s := nops(fL);
    for i from 1 to s do a[i] := 0 od;
    (p,r) := (f,0);
    while (p <> 0) do
      if (verb) then print('p' = p) fi;
      i := 1;
      while (i <= s) and not(monom_divis( LT(p,X), LT(fL[i],X),
  X )) do i := i+1 od;
                                                  Save these.
      if (i > s) then
        (p,r) := (p-LT(p,X),r+LT(p,X));
        if (verb) then print('r' = r) fi;
      else
        t := LT(p,X)/LT(fL[i],X);
        (p,a[i]) := (p-expand(t*fL[i]),a[i]+t);
        if (verb) then print(`a[`||i||`]` = a[i]) fi;
      fi;
    od;
    return (seq(a[i], i=1..s),r);      [seq(a[i], i=1..s)], r   better.
  end:
```

We write a procedure to compute leading terms with respect to lexicographic order.

```
> LTlex := proc( f, X::list ) local c, m;
    c := lcoeff( f, X, 'm' );
    return c*m;
  end:
```

We write a procedure to compute leading terms with respect to graded lexicographic order.

```
> LTgrlex := proc( f, X::list ) local d, g, t;
    if not(type( f, `+` )) then return f fi;
    d := max( seq( degree(t), t=f ) );
    g := add( `if`( degree(t) = d, t, 0 ), t=f );
    return LTlex(g,X);
  end:
```

We write a procedure to compute the multidegree of a polynomial f in variables X. The multidegree is dependent on the monomial ordering chosen, so as in the division algorithm we input a procedure to compute leading terms.

```
> multideg := proc( f, X::list, LT::procedure )
    if type( f, `+` ) then return multideg( LT(f,X), X, LT )
  fi;
    return map2( degree, f, X );
  end:
```

We write a procedure to test if one monomial term m_1 is divisible by another monomial term m_2 in variables X. Note that the property of divisibility is independent of the monomial ordering chosen.

```
> monom_divis := proc( m1, m2, X::list )
  local a, i;
  global multideg;
    if type( m1, `+` ) or type( m2, `+` ) then error "inputs
  should be monomials" fi;
    a := multideg( m1, X ) - multideg( m2, X );
    for i in a do if (i < 0) then return false fi od;
    return true;
  end:
```

3.1

For the first part we enter f to divide and the divisors f_1, f_2 from Exercise 1.

```
> f := x^7*y^2+x^3*y^2-y+1;
  (f1,f2) := (x*y^2-x,x-y^3);
```

$$f := x^7 y^2 + x^3 y^2 - y + 1$$

$$f1, f2 := x y^2 - x, x - y^3$$

We compute the remainder of f divided by $[f_1, f_2]$ using the grlex order, first with Maple's remainder procedure in the Groebner package, then using our own implementation of the division algorithm. We then check that $f = a_1 f_1 + a_2 f_2 + r$ from our procedure, and that our remainder is the same as the one Maple computed.

```
> r_ := Groebner[NormalForm]( f, [f1,f2], grlex(op(X)) );
```

$$r\_ := x^7 + x^3 - y + 1$$

```
> (a1,a2,r) := DIVIDE( f, [f1,f2], X, LTgrlex );
  f - (expand(a1*f1)+expand(a2*f2)+r), r_-r;
```

$$a1, a2, r := x^6 + x^2, 0, x^7 + x^3 - y + 1$$

$$0, 0$$

We repeat for the remainder of f divided by $[f_1, f_2]$ using the lex order.

```
> r_ := Groebner[NormalForm]( f, [f1,f2], plex(op(X)) );
```

$$r\_ := -y + 1 + 2\,y^3$$

```
> (a1,a2,r) := DIVIDE( f, [f1,f2], X, LTlex );
  f - (expand(a1*f1)+expand(a2*f2)+r), r_-r;
```

$$a1, a2, r := x^6 + x^5\,y + x^4\,y^2 + x^4 + x^3\,y + x^2\,y^2 + 2\,x^2 + 2\,x\,y + 2\,y^2 + 2,$$
$$x^6 + x^5\,y + x^4 + x^3\,y + 2\,x^2 + 2\,x\,y + 2, -y + 1 + 2\,y^3$$

$$0, 0$$

We repeat for the remainder of f divided by $[f_2, f_1]$ using the grlex order.

```
> r_ := Groebner[NormalForm]( f, [f2,f1], grlex(op(X)) );
```

$$r\_ := x^7 + x^3 - y + 1$$

```
> (a2,a1,r) := DIVIDE( f, [f2,f1], X, LTgrlex );
  f - (expand(a1*f1)+expand(a2*f2)+r), r_-r;
```

$$a2, a1, r := 0, x^6 + x^2, x^7 + x^3 - y + 1$$

$$0, 0$$

We repeat for the remainder of f divided by $[f_2, f_1]$ using the lex order.

```
> r_ := Groebner[NormalForm]( f, [f2,f1], plex(op(X)) );
```

$$r\_ := -y + 1 + y^{23} + y^{11}$$

```
> (a2,a1,r) := DIVIDE( f, [f2,f1], X, LTlex );
  f - (expand(a1*f1)+expand(a2*f2)+r), r_-r;
```

$$a2, a1, r :=$$
$$x^6\,y^2 + x^5\,y^5 + x^4\,y^8 + x^3\,y^{11} + x^2\,y^{14} + x^2\,y^2 + x\,y^{17} + x\,y^5 + y^{20} + y^8, 0, -y + 1 + y^{23} + y^{11}$$

$$0, 0$$

In all cases, the results match what we computed by hand. The remainder is dependent on both the monomial ordering and the order of the divisors.

3.2

For the first part we enter f to divide and the divisors f_1, f_2, f_3 from Exercise 2.

```
> f := x*y^2*z^2+x*y-y*z;
  (f1,f2,f3) := (x-y^2,y-z^3,z^2-1);
```

$$f := x\,y^2\,z^2 + x\,y - y\,z$$

$$f1, f2, f3 := x - y^2, y - z^3, z^2 - 1$$

We compute the remainder of f divided by $[f_1, f_2, f_3]$ using the grlex order, first with Maple's remainder procedure in the Groebner package, then using our own implementation of the division algorithm. We then check that $f = a_1 f_1 + a_2 f_2 + a_3 f_3 + r$ from our procedure, and that our remainder is the same as the one Maple computed. We then repeat using the lex order.

```
> r_  := Groebner[NormalForm]( f,  [f1,f2,f3], grlex(op(X)) );
  (a1,a2,a3,r)  := DIVIDE( f,  [f1,f2,f3], X, LTgrlex );
  f - (expand(a1*f1)+expand(a2*f2)+expand(a3*f3)+r), r_-r;
  r_  := Groebner[NormalForm]( f,  [f1,f2,f3], plex(op(X)) );
  (a1,a2,a3,r)  := DIVIDE( f,  [f1,f2,f3], X, LTlex );
  f - (expand(a1*f1)+expand(a2*f2)+expand(a3*f3)+r), r_-r;
```

$$r\_ := x\,y - y\,z + x^2$$

$$a1, a2, a3, r := -x\,z^2, 0, x^2, x\,y - y\,z + x^2$$

$$0, 0$$

$$r\_ := z$$

$$a1, a2, a3, r := y^2 z^2 + y, y^3 z^2 + y^2 z^5 + y^2 + y\,z^8 + y\,z^3 + z^{11} + z^6 - z,$$
$$z^{12} + z^{10} + z^8 + z^7 + z^6 + z^5 + z^4 + z^3 + z, z$$

$$0, 0$$

We repeat for the remainder of f divided by $[f_2, f_3, f_1]$ using the grlex order, then the lex order.

```
> r_  := Groebner[NormalForm]( f,  [f2,f3,f1], grlex(op(X)) );
  (a2,a3,a1,r)  := DIVIDE( f,  [f2,f3,f1], X, LTgrlex );
  f - (expand(a1*f1)+expand(a2*f2)+expand(a3*f3)+r), r_-r;
  r_  := Groebner[NormalForm]( f,  [f2,f3,f1], plex(op(X)) );
  (a2,a3,a1,r)  := DIVIDE( f,  [f2,f3,f1], X, LTlex );
  f - (expand(a1*f1)+expand(a2*f2)+expand(a3*f3)+r), r_-r;
```

$$r\_ := x\,y - y\,z + x^2$$

$$a2, a3, a1, r := 0, x\,y^2, -x, x\,y - y\,z + x^2$$

$$0, 0$$

$$r\_ := z$$

$$a2, a3, a1, r :=$$
$$x\,y\,z^2 + x\,z^5 + x + y\,z + y + z^4 + z^3 - z, x\,z^6 + x\,z^4 + x\,z^2 + x\,z + x + z^5 + z^4 + z^3 + z, z + 1, z$$

$$0, 0$$

We repeat for the remainder of f divided by $[f_3, f_1, f_2]$ using the grlex order, then the lex order.

```
> r_  := Groebner[NormalForm]( f,  [f3,f1,f2], grlex(op(X)) );
  (a3,a1,a2,r)  := DIVIDE( f,  [f3,f1,f2], X, LTgrlex );
  f - (expand(a1*f1)+expand(a2*f2)+expand(a3*f3)+r), r_-r;
  r_  := Groebner[NormalForm]( f,  [f3,f1,f2], plex(op(X)) );
  (a3,a1,a2,r)  := DIVIDE( f,  [f3,f1,f2], X, LTlex );
  f - (expand(a1*f1)+expand(a2*f2)+expand(a3*f3)+r), r_-r;
```

$$r\_ := x\,y - y\,z + x^2$$

$$a3, a1, a2, r := x y^2, -x, 0, x y - y z + x^2$$

$$0, 0$$

$$r\_ := z$$

$a3, a1, a2, r :=$

$$x y^2 + y^3 z + y^2 z^2 + z y^2 + y^2 + y z^2 + y z + y + z, y + y^2, y^3 + z y^2 + y^2 + y z + y + 1, z$$

$$0, 0$$

In this example, the remainders were independent of the order of the divisors, but still dependent on the monomial ordering. ✓

2.3.5

We enter f and the divisors f_1, f_2. We will use grlex order throughout this exercise.

```
> f := x^3-x^2*y-x^2*z+x;
  (f1,f2) := (x^2*y-z,x*y-1);
```

$$f := x^3 - x^2 y - x^2 z + x$$

$$f1, f2 := x^2 y - z, x y - 1$$

a)

We compute the remainders of f on division by $[f_1, f_2]$ and $[f_2, f_1]$.

```
> r1 := DIVIDE( f, [f1,f2], X, LTgrlex, verbose=true )[-1];
```

$$p = x^3 - x^2 y - x^2 z + x$$

$$r = x^3 \quad ✓$$

$$p = -x^2 y - x^2 z + x$$

$$a[1] = -1 \quad ✓$$

$$p = -x^2 z + x - z$$

$$r = x^3 - x^2 z \quad ✓$$

$$p = x - z$$

$$r = x^3 - x^2 z + x$$

$$p = -z$$

$$r = x^3 - x^2 z + x - z$$

$$r1 := x^3 - x^2 z + x - z \quad ✓$$

```
> r2 := DIVIDE( f, [f2,f1], X, LTgrlex, verbose=true )[-1];
```

$$p = x^3 - x^2 y - x^2 z + x$$

$$r = x^3 \quad ✓$$

$$p = -x^2 y - x^2 z + x$$

$$a[1] = -x \quad ✓$$

$$p = -x^2 z$$

$$r = x^3 - x^2 z$$

$$r2 := x^3 - x^2 z$$ ✓

We see that r_1 has two additional terms. In the second iteration of the main "while" loop in the division algorithm we compute different terms to add to the quotients in the two divisions. When dividing by $[f_1, f_2]$ we are then left with two additional terms in p that aren't there when dividing by $[f_2, f_1]$.

b)

```
> r := r1-r2;
  f1+expand(-x*f2);
```

$$r := x - z$$ ✓

$$x - z$$

$r = r_1 - r_2$ is in the ideal generated by $\{f_1, f_2\}$, since $r = f_1 + (-x) f_2$.

c)

The remainder of r on division by $[f_1, f_2]$ is r itself, ✓ because no term in r is divisible by LT(f_1) ✓ or LT(f_2) (that is the condition satisfied by remainders r_1 and r_2). ✓

```
> DIVIDE( r, [f1,f2], X, LTgrlex )[-1];
```

$$x - z$$

d)

```
> g := expand(-y*f1)+expand((x*y+1)*f2);
  DIVIDE( g, [f1,f2], X, LTgrlex )[-1];
```

$$g := y z - 1$$

$$y z - 1$$ ✓

$g = (-y) f_1 + (x y + 1) f_2$ is another polynomial in the ideal generated by $\{f_1, f_2\}$ whose remainder on division by $[f_1, f_2]$ is nonzero. In fact, the remainder is g itself.

e)

The division algorithm does not give us a solution for the ideal membership problem for the ideal generated by $\{f_1, f_2\}$. If the division algorithm returns a zero remainder for f then f must be in the ideal. However if the algorithm returns a nonzero remainder for f then f may or may not be in the ideal. ✓

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>

MATH 800 Assignment 2

(due June 7, 2006, 9:30)

This worksheet uses the division algorithm components from the Section 2.3 worksheet.

2.6.2

Divide $x\,y$ by the Groebner basis (for lex order) $\{x+z, x-z\}$ in both possible orders of the divisors.

```
> (f1,f2) := (x+z,y-z);
```

$$f1, f2 := x+z, y-z$$

```
> DIVIDE( x*y, [f1,f2], X, LTlex );
```

$$y, -z, -z^2$$

```
> DIVIDE( x*y, [f2,f1], X, LTlex );
```

$$x, z, -z^2$$

The remainders are the same (as they must be), but the quotients are different. Hence uniqueness of the remainder is the best we can hope for, even when dividing by a Groebner basis.

2.6.9

We begin by programming a procedure to calculate S-polynomials. This procedure computes $S(f, g)$ with respect to the monomial order defined by the variable ordering X, and the leading-term procedure $LT(g, X)$. If the optional parameter *fractionfree* = *true* is given then the resulting S-polynomial will be scaled so that none of its coefficients are fractions.

```
> S_poly := proc( f, g, X::list, LT ) local ltf, ltg, a, b, c,
  lcm, i, fractfree;
    fractfree := false;          if hasoption( [args[5..-1]], fractionfree, fractfree) then ...
    if (nargs > 4) then for c in args[5..-1] do
      if (op(1,c) = fractionfree) then fractfree := op(2,c) fi;
    od fi;
    (ltf,ltg) := (LT(f,X),LT(g,X));
    (a,b) := multideg( ltf, X ),multideg( ltg, X );
    lcm := mul( X[i]^max(a[i],b[i]), i=1..nops(a) );
    if (fractfree) then
      c := ilcm( lcoeff( ltf, X ), lcoeff( ltg, X ) );
      lcm := c*lcm;
    fi;
    return expand(lcm/ltf*f)-expand(lcm/ltg*g);
  end:
```

Now we will determine whether the following sets G are Groebner bases for the ideals they generate, by testing whether the remainders of the S-polynomials $S(g_i, g_j)$, on division by G, are all zero. If this is the case, then by Theorem 6 G is a Groebner basis.

a)

```
> G := [x^2-y,x^3-z];
  for i from 1 to nops(G) do for j from i+1 to nops(G) do
    S := S_poly( G[i], G[j], X, LTgrlex );
    Sg := DIVIDE( S, G, X, LTgrlex )[-1];
    print(S, Sg);
  od od;
```

$$G := [x^2 - y, x^3 - z]$$
$$-x\,y + z, -x\,y + z$$

The remainder of the S-polynomial is not zero, so G is not a Groebner basis with respect to grlex order.

b)

```
> G := [x^2-y,x^3-z];
  for i from 1 to nops(G) do for j from i+1 to nops(G) do
    S := S_poly( G[i], G[j], X, LTlex );
    Sg := DIVIDE( S, G, X, LTlex )[-1];
    print(S, Sg);
  od od;
```

$$G := [x^2 - y, x^3 - z]$$
$$-x\,y + z, -x\,y + z$$

The remainder of the S-polynomial is not zero, so G is not a Groebner basis with respect to lex order.

c)

```
> G := [x*y^2-x*z+y,x*y-z^2,x-y*z^4];
  for i from 1 to nops(G) do for j from i+1 to nops(G) do
    S := S_poly( G[i], G[j], X, LTlex );
    Sg := DIVIDE( S, G, X, LTlex )[-1];
    print(S, Sg);
  od od;
```

$$G := [x\,y^2 - x\,z + y, x\,y - z^2, x - y\,z^4]$$
$$-x\,z + y + y\,z^2, y + y\,z^2 - y\,z^5$$
$$-x\,z + y + y^3\,z^4, y + y^3\,z^4 - y\,z^5$$
$$-z^2 + z^4\,y^2, -z^2 + z^4\,y^2$$

The remainders of the S-polynomials are not all zero, so G is not a Groebner basis with respect to lex order.

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```
>
```

MATH 800 Assignment 2

(due June 7, 2006, 9:30)

This worksheet uses the division algorithm components from the Section 2.3 worksheet.

2.6.2

Divide $x\,y$ by the Groebner basis (for lex order) $\{x+z, y-z\}$ in both possible orders of the divisors.

```
> (f1,f2) := (x+z,y-z);
```

$$f1, f2 := x+z, y-z$$

```
> DIVIDE( x*y, [f1,f2], X, LTlex );
```

$$y, -z, -z^2$$

```
> DIVIDE( x*y, [f2,f1], X, LTlex );
```

$$x, z, -z^2$$

The remainders are the same (as they must be), but the quotients are different. Hence uniqueness of the remainder is the best we can hope for, even when dividing by a Groebner basis.

2.6.9

We begin by programming a procedure to calculate S-polynomials. This procedure computes $S(f, g)$ with respect to the monomial order defined by the variable ordering X, and the leading-term procedure $LT(g, X)$. If the optional parameter *fractionfree* = *true* is given then the [illegible]

```
> S_poly := proc( f, g, X::list, LT ) local ltf, ltg, a, b, c,
  lcm, i, fractfree;
    fractfree := false;
    if (nargs > 4) then for c in args[5..-1] do
      if (op(1,c) = fractionfree) then fractfree := op(2,c) fi;
    od fi;
    (ltf,ltg) := (LT(f,X),LT(g,X));
    (a,b) := multideg( ltf, X ),multideg( ltg, X );
    lcm := mul( X[i]^max(a[i],b[i]), i=1..nops(a) );
    if (fractfree) then
      c := ilcm( lcoeff( ltf, X ), lcoeff( ltg, X ) );
      lcm := c*lcm;
    fi;
    return expand(lcm/ltf*f)-expand(lcm/ltg*g);
  end:
```

[illegible]

```
> G := [x^2-y,x^3-z];
  for i from 1 to nops(G) do for j from i+1 to nops(G) do
    S := S_poly( G[i], G[j], X, LTgrlex );
    Sg := DIVIDE( S, G, X, LTgrlex )[-1];
    print(S, Sg);
  od od;
```

$$G := [x^2 - y, x^3 - z]$$

$$-x\,y + z, -x\,y + z$$

The remainder of the S-polynomial is not zero, so G is not a Groebner basis with respect to grlex order.

b)

```
> G := [x^2-y,x^3-z];
  for i from 1 to nops(G) do for j from i+1 to nops(G) do
    S := S_poly( G[i], G[j], X, LTlex );
    Sg := DIVIDE( S, G, X, LTlex )[-1];
    print(S, Sg);
  od od;
```

$$G := [x^2 - y, x^3 - z]$$

$$-x\,y + z, -x\,y + z$$

The remainder of the S-polynomial is not zero, so G is not a Groebner basis with respect to lex order.

c)

```
> G := [x*y^2-x*z+y,x*y-z^2,x-y*z^4];
  for i from 1 to nops(G) do for j from i+1 to nops(G) do
    S := S_poly( G[i], G[j], X, LTlex );
    Sg := DIVIDE( S, G, X, LTlex )[-1];
    print(S, Sg);
  od od;
```

$$G := [x\,y^2 - x\,z + y, x\,y - z^2, x - y\,z^4]$$

$$-x\,z + y + y\,z^2, y + y\,z^2 - y\,z^5$$

$$-x\,z + y + y^3\,z^4, y + y^3\,z^4 - y\,z^5$$

$$-z^2 + z^4\,y^2, -z^2 + z^4\,y^2$$

The remainders of the S-polynomials are not all zero, so G is not a Groebner basis with respect to lex order.

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>

MATH 800 Assigment 2
(due June 7, 2006, 9:30)
This worksheet uses the some of the division algorithm components from the Section 2.3 worksheet.

Additional Problem 1

We define procedures that take in two monomial terms m_1, m_2 and a variable ordering X, that return true if and only if $m_1 < m_2$ in the graded lex order or graded reverse lex order, respectively, defined by the variable ordering.

```
> MonOrder_grlex := proc( m1, m2, X::list ) local a, b, tda, tdb,
  i; global multideg;
    (a,b) := (multideg( m1, X ),multideg( m2, X ));
    (tda,tdb) := (add( i, i=a ),add( i, i=b ));
    if (tda < tdb) then return true;
    elif (tda > tdb) then return false fi;
    a := b-a;
    for i from 1 to nops(a) do
      if (a[i] > 0) then return true;
      elif (a[i] < 0) then return false fi;
    od;
    return false;
  end:
> MonOrder_grevlex := proc( m1, m2, X::list ) local a, b, tda,
  tdb, i; global multideg;
    (a,b) := (multideg( m1, X ),multideg( m2, X ));
    (tda,tdb) := (add( i, i=a ),add( i, i=b ));
    if (tda < tdb) then return true;
    elif (tda > tdb) then return false fi;
    a := b-a;
    for i from 1 to nops(a) do
      if (a[-i] < 0) then return true;
      elif (a[-i] > 0) then return false fi;
    od;
    return false;
  end:
```

Now we can generate all monomials of total degree less than or equal to 3, for $k[x, y]$ and $k[x, y, z]$, and sort them into grlex order and into grevlex order.

```
> mon2var := [seq( seq( x^i*y^(j-i), j=i..3 ), i=0..3 )];
  mon3var := [seq( seq( seq( x^i*y^j*z^(k-j-i), k=i+j..3 ),
  j=0..3 ), i=0..3 )];
```

✓

$$mon2var := [1, y, y^2, y^3, x, x\,y, x\,y^2, x^2, x^2\,y, x^3]$$

$$mon3var := [1, z, z^2, z^3, y, y\,z, y\,z^2, y^2, z\,y^2, y^3, x, x\,z, x\,z^2, x\,y, x\,y\,z, x\,y^2, x^2, x^2\,z, x^2\,y, x^3]$$

In $k[x, y]$:

```
> sort( mon2var,  (m,n) -> MonOrder_grlex( m,  n,  [x,y] ) );
  sort( mon2var,  (m,n) -> MonOrder_grevlex( m,  n,  [x,y] ) );
```

$$[1, y, x, y^2, x\,y, x^2, y^3, x\,y^2, x^2\,y, x^3]$$
$$[1, y, x, y^2, x\,y, x^2, y^3, x\,y^2, x^2\,y, x^3]$$

We see that in two variables, there is no difference between grlex and grevlex order. This can be explained as follows: If we have two monomials $x^a\,y^b$ and $x^c\,y^d$ of equal total degree ($a+b=c+d$), then the conditions $c<a$ (so $x^c\,y^d<x^a\,y^b$ in grlex order) and $b<d$ (so $x^c\,y^d<x^a\,y^b$ in grevlex order) are equivalent.

We can check that our monomial order comparison functions are correct by sorting the same list using Maple's Groebner package monomial ordering functions.

```
> sort( mon2var,  (m,n) -> Groebner[TestOrder]( m,  n,  grlex(x,y) )
  );
  sort( mon2var,  (m,n) -> Groebner[TestOrder]( m,  n,  tdeg(x,y) )
  );
```

$$[1, y, x, y^2, x\,y, x^2, y^3, x\,y^2, x^2\,y, x^3]$$
$$[1, y, x, y^2, x\,y, x^2, y^3, x\,y^2, x^2\,y, x^3]$$

In $k[x, y, z]$:

```
> sort( mon3var,  (m,n) -> MonOrder_grlex( m,  n,  X ) );
  sort( mon3var,  (m,n) -> MonOrder_grevlex( m,  n,  X ) );
```

$$[1, z, y, x, z^2, y\,z, y^2, x\,z, x\,y, x^2, z^3, y\,z^2, z\,y^2, y^3, x\,z^2, x\,y\,z, x\,y^2, x^2\,z, x^2\,y, x^3]$$
$$[1, z, y, x, z^2, y\,z, x\,z, y^2, x\,y, x^2, z^3, y\,z^2, x\,z^2, z\,y^2, x\,y\,z, x^2\,z, y^3, x\,y^2, x^2\,y, x^3]$$

In three variables, the difference between grlex and grevlex order can be described as follows: for monomials of the same total degree, grlex orders those with the most x's first, while grevlex orders those with the least z's first.

We can check that our monomial order comparison functions are correct by sorting the same list using Maple's Groebner package monomial ordering functions.

```
> sort( mon3var,  (m,n) -> Groebner[TestOrder]( m,  n,  grlex(op(X))
  ) );
  sort( mon3var,  (m,n) -> Groebner[TestOrder]( m,  n,  tdeg(op(X))
  ) );
```

$$[1, z, y, x, z^2, y\,z, y^2, x\,z, x\,y, x^2, z^3, y\,z^2, z\,y^2, y^3, x\,z^2, x\,y\,z, x\,y^2, x^2\,z, x^2\,y, x^3]$$
$$[1, z, y, x, z^2, y\,z, x\,z, y^2, x\,y, x^2, z^3, y\,z^2, x\,z^2, z\,y^2, x\,y\,z, x^2\,z, y^3, x\,y^2, x^2\,y, x^3]$$

To further describe grevlex order, notice that among monomials of the same total degree, grevlex order with $x>y>z$ is the reverse of lex order with $z>y>x$.

```
> mon3var3deg := [seq( seq( x^i*y^j*z^(3-j-i), j=0..3-i ), i=0..3
  )]:
  sort( mon3var3deg,  (m,n) -> Groebner[TestOrder]( m,  n,
  tdeg(op(X)) ) );
```

MATH 800 Assignment 2
(due June 7, 2006, 9:30)
This worksheet uses the some of the division algorithm components from the Section 2.3 worksheet.

Additional Problem 1

We define procedures that take in two monomial terms m_1, m_2 and a variable ordering X, that return true if and only if $m_1 < m_2$ in the graded lex order or graded reverse lex order, respectively, defined by the variable ordering.

```
> MonOrder_grlex := proc( m1, m2, X::list ) local a, b, tda, tdb,
  i; global multideg;
    (a,b) := (multideg( m1, X ),multideg( m2, X ));
    (tda,tdb) := (add( i, i=a ),add( i, i=b ));
    if (tda < tdb) then return true;
    elif (tda > tdb) then return false fi;
    a := b-a;
    for i from 1 to nops(a) do
      if (a[i] > 0) then return true;
      elif (a[i] < 0) then return false fi;
    od;
    return false;
  end:
> MonOrder_grevlex := proc( m1, m2, X::list ) local a, b, tda,
  tdb, i; global multideg;
    (a,b) := (multideg( m1, X ),multideg( m2, X ));
    (tda,tdb) := (add( i, i=a ),add( i, i=b ));
    if (tda < tdb) then return true;
    elif (tda > tdb) then return false fi;
    a := b-a;
    for i from 1 to nops(a) do
      if (a[-i] < 0) then return true;
      elif (a[-i] > 0) then return false fi;
    od;
    return false;
  end:
```

Now we can generate all monomials of total degree less than or equal to 3, for $k[x, y]$ and $k[x, y, z]$, and sort them into grlex order and into grevlex order.

```
> mon2var := [seq( seq( x^i*y^(j-i), j=i..3 ), i=0..3 )];
  mon3var := [seq( seq( seq( x^i*y^j*z^(k-j-i), k=i+j..3 ),
  j=0..3 ), i=0..3 )];
```

$$mon2var := [1, y, y^2, y^3, x, x\,y, x\,y^2, x^2, x^2\,y, x^3]$$

✓

$$mon3var := [1, z, z^2, z^3, y, y\,z, y\,z^2, y^2, z\,y^2, y^3, x, x\,z, x\,z^2, x\,y, x\,y\,z, x\,y^2, x^2, x^2\,z, x^2\,y, x^3]$$

```
sort( mon3var3deg, (m,n) -> Groebner[TestOrder]( n, m,
plex(z,y,x) ) );
```

$$[z^3, y z^2, x z^2, z y^2, x y z, x^2 z, y^3, x y^2, x^2 y, x^3]$$

$$[z^3, y z^2, x z^2, z y^2, x y z, x^2 z, y^3, x y^2, x^2 y, x^3]$$

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