Scott Cowan
200034814

Model Solutions.

MATH 800 Assignment #6

7 Aug. 2006

5.2.3. Prove Theorem 6: Let I be an ideal in $k[x_1,\dots,x_n]$. The quotient $k[x_1,\dots,x_n]/I$ is a commutative ring under the given sum and product operations.

Proof: By the definition of $+$ and $\cdot$, $k[x_1,\dots,x_n]$ is closed under these operations, and by Proposition 5, they are well-defined.

Let $[f],[g],[h] \in k[x_1,\dots,x_n]/I$. Then

$[f]+[g]$ and $[f]\cdot[g]$

$=[f+g]$ $\quad$ $=[f\cdot g]$ by def. of $+,\cdot$ in $k[x_1,\dots,x_n]/I$

$=[g+f]$ $\quad$ $=[g\cdot f]$ ✓ by commutative properties of $k[x_1,\dots,x_n]$

$=[g]+[f]$ ✓ $\quad$ $=[g]\cdot[f]$ ✓ by def. of $+,\cdot$ in $k[x_1,\dots,x_n]/I$.

So $+$ and $\cdot$ are commutative in $k[x_1,\dots,x_n]/I$. Furthermore

$([f]+[g])+[h]$ and $([f]\cdot[g])\cdot[h]$

$=[f+g]+[h]$ $\quad$ $=[f\cdot g]\cdot[h]$ $\Big\}$ by def. of $+,\cdot$ in $k[x_1,\dots,x_n]/I$

$=[(f+g)+h]$ $\quad$ $=[(f\cdot g)\cdot h]$

$=[f+(g+h)]$ $\quad$ $=[f\cdot(g\cdot h)]$ by associative ✓ properties of $k[x_1,\dots,x_n]$

$=[f]+[g+h]$ $\quad$ $=[f]\cdot[g\cdot h]$ $\Big\}$ by def. of $+,\cdot$ in $k[x_1,\dots,x_n]/I$.

$=[f]+([g]+[h])$ $\quad$ $=[f]\cdot([g]\cdot[h])$

So $+$ and $\cdot$ are associative in $k[x_1,\dots,x_n]/I$. Also

$[f]\cdot([g]+[h])$

$=[f]\cdot[g+h]$ by def. of $+$ in $k[x_1,\dots,x_n]/I$

$=[f\cdot(g+h)]$ by def. of $\cdot$ in $k[x_1,\dots,x_n]/I$

$=[f\cdot g+f\cdot h]$ by distributive property of $k[x_1,\dots,x_n]$

$=[f\cdot g]+[f\cdot h]$ by def. of $+$ in $k[x_1,\dots,x_n]/I$

$=[f]\cdot[g]+[f]\cdot[h]$ ✓ by def. of $\cdot$ in $k[x_1,\dots,x_n]/I$.

So $\cdot$ is distributive over $+$ in $k[x_1,\dots,x_n]/I$.

Now consider $[0],[1] \in k[x_1,\dots,x_n]/I$ (where $0,1 \in k[x_1,\dots,x_n]$).
$\forall\, [f] \in k[x_1,\dots,x_n]/I$ we have

$[f]+[0]$ and $[f]\cdot[1]$

$=[f+0]$ $\quad$ $=[f\cdot 1]$ by def. of $+,\cdot$ in $k[x_1,\dots,x_n]/I$

$=[f]$ ✓ $\quad$ $=[f]$ by 0,1 identities in $k[x_1,\dots,x_n]$

so [0] and [1] are the additive and multiplicative identities
✓ (respectively) in $k[x_1,\dots,x_n]/I$.

Finally, $\forall [f] \in k[x_1,\dots,x_n]/I$ we have $[-f] \in k[x_1,\dots,x_n]/I$, and

$[f]+[-f]$

$= [f+(-f)]$ by def. of $+$ in $k[x_1,\dots,x_n]/I$

$= [0]$ by additive inverse property of $k[x_1,\dots,x_n]$.

So every $[f] \in k[x_1,\dots,x_n]/I$ has an additive inverse (negative) $[-f]$ in $k[x_1,\dots,x_n]/I$.

6 $\therefore k[x_1,\dots,x_n]/I$ is a commutative ring (with identity) under the sum and product operations. □

5.2.6. Show that $\mathbb{R}[x]/\langle x^2-4x+3\rangle$ is not an integral domain.

Let $f=[x-1], g=[x-3] \in \mathbb{R}[x]/\langle x^2-4x+3\rangle$.
If $f=[0]$ in $\mathbb{R}[x]/\langle x^2-4x+3\rangle$ then $x-1 \equiv 0 \mod \langle x^2-4x+3\rangle$
so $(x-1)-0 = x-1 \in \langle x^2-4x+3\rangle$. But any element in $\langle x^2-4x+3\rangle$ is of the form $h\cdot(x^2-4x+3)$ $(h \in \mathbb{R}[x])$ and so is 0 or has
4 OK degree ≥ 2. Thus $x-1 \notin I$.
So $f \neq [0]$, and $g \neq [0]$ by the same argument.
However $f\cdot g = [x-1]\cdot[x-3] = [(x-1)\cdot(x-3)] = [x^2-4x+3]$, and
$x^2-4x+3 = (x^2-4x+3)-0 \in I \Rightarrow x^2-4x+3 \equiv 0 \mod I \Rightarrow [x^2-4x+3]=[0]$.
$f\cdot g = [0]$, so f and g are zero-divisors in $\mathbb{R}[x]/\langle x^2-4x+3\rangle$. ✓
$\therefore \mathbb{R}[x]/\langle x^2-4x+3\rangle$ is not an integral domain.

* 5.2.14. a) Let $I = \langle x^3-x\rangle \in R = \mathbb{R}[x]$. Determine the ideals in the quotient ring R/I, and draw a containment diagram.

By Proposition 10, there is a one-to-one correspondence between the ideals in R/I and the ideals in R containing I.
$R = \mathbb{R}[x]$ is a principal ideal domain, so the ideals in R containing I are precisely those generated by divisors of x^3-x. ✓
$x^3-x = x(x^2-1) = x(x-1)(x+1)$,

so the divisors of x^3-x are all products of zero, one, two, or all three of the factors x, $x-1$ and $x+1$:

$1,\ x,\ x-1,\ x+1,\ x(x-1)=x^2-x,\ x(x+1)=x^2+x,\ (x-1)(x+1)=x^2-1,$ and $x(x-1)(x+1)=x^3-x$.

Thus we have

ideals in R containing I	corresponding ideals in R/I
$\langle 1\rangle = R$	$\langle [1]\rangle = R/I$
$\langle x\rangle$	$\langle [x]\rangle$
$\langle x-1\rangle$	$\langle [x-1]\rangle$
$\langle x+1\rangle$	$\langle [x+1]\rangle$
$\langle x^2-x\rangle$	$\langle [x^2-x]\rangle$
$\langle x^2+x\rangle$	$\langle [x^2+x]\rangle$
$\langle x^2-1\rangle$	$\langle [x^2-1]\rangle$
$\langle x^3-x\rangle = I$	$\langle [x^3-x]\rangle = \{[0]\}$.

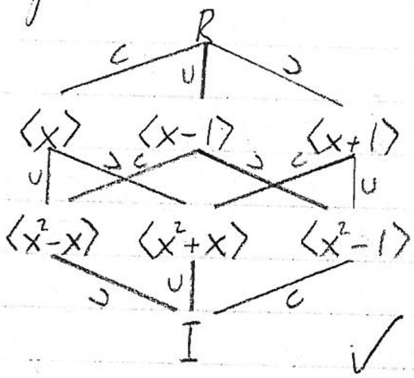
Now for $f,g\in R$ we have $g\mid f \iff \langle f\rangle \subset \langle g\rangle$ (since $f=h\cdot g$).

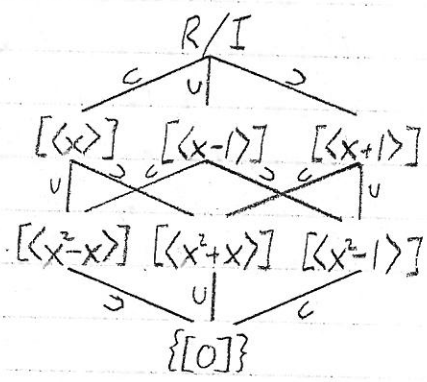
5

Also $\langle f\rangle \subset \langle g\rangle \Rightarrow \langle [f]\rangle \subset \langle [g]\rangle$ because if $f=h\cdot g$ then $\forall\, [p][f]\in\langle [f]\rangle$ we have

$[p]\cdot[f]=[pf]=[phg]=[ph]\cdot[g]\in\langle [g]\rangle$.

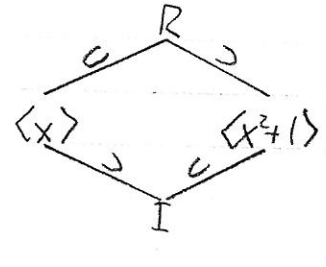
This gives us

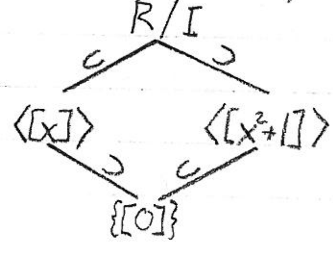




b) How does the answer change if $I=\langle x^3+x\rangle$?

$x^3+x = x(x^2+1)$ where x^2+1 is irreducible in $R=\mathbb{R}[x]$, so we get





5.3.1. Prove Proposition 1(ii): The elements of $\{x^\alpha : x^\alpha \notin \langle LT(I) \rangle\}$ are linearly independent modulo I.

with

Proof: Suppose $\sum_\alpha c_\alpha x^\alpha \equiv 0$ mod I, where each $x^\alpha \notin \langle LT(I)\rangle$, $c_\alpha \neq 0$

Then $\sum_\alpha c_\alpha x^\alpha - 0 \in I$

4/5

Not Very Good,

$\Rightarrow \sum_\alpha c_\alpha x^\alpha \in I$

$\Rightarrow LT(\sum_\alpha c_\alpha x^\alpha) \in LT(I) \subset \langle LT(I)\rangle$.

But no term of $\sum_\alpha c_\alpha x^\alpha \in \langle LT(I)\rangle$, so there must in fact be no terms in $\sum_\alpha c_\alpha x^\alpha$, ie. $\sum_\alpha c_\alpha x^\alpha = 0$

$\Rightarrow$ all $c_\alpha = 0$. □

9 5.3.5. [See Maple attachment]

5.3.8. Prove Corollary 7: Let $I \subset \mathbb{C}[x_1, \ldots, x_n]$ be an ideal such that for each i, some power $x_i^{m_i} \in \langle LT(I)\rangle$. Then $|\mathbb{V}(I)| \leq \prod m_i$.

Proof: For each i, $1 \leq i \leq n$, there is some $m_i \geq 0$ s.t. $x_i^{m_i} \in \langle LT(I)\rangle$. By Theorem 6, this means that in any Groebner basis for I, for each i there is a polynomial g where $LT(g)$ is a power of x_i.

5 X

So for each i, $1 \leq i \leq n$, let G_i be a reduced GB for I w.r.t. lexicographic order with $x_j > x_i$ $\forall j \neq i$.

Then in each G_i there is a polynomial g_i with $LT(g_i) = x_i^{t_i}$ for some $t_i \geq 0$. Now since G_i is reduced, no other polynomial in G_i has a leading term that is a power of x_i. ✓ Then it must be the case that $LT(g_i) | x_i^{m_i}$ so $p_i \leq m_i$. ✓ Furthermore, since x_i is the least variable in the monomial ordering for G_i, g_i must be a polynomial in x_i alone. ✓ So there are at most $p_i \leq m_i$ values for x_i s.t. g_i vanishes, and any point in $\mathbb{V}(I)$ must have one of these values as its i^{th} coordinate.

✓ ✓ X Can't conclude this.

Proof is wrong.

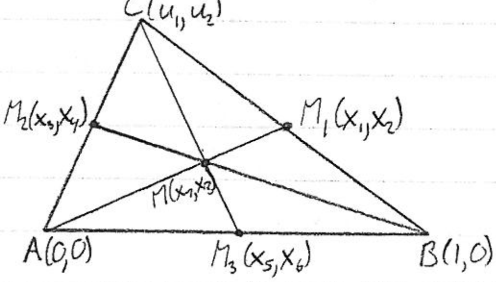
$\therefore$ for each i, $1 \leq i \leq n$, there are $\leq m_i$ choices for the i^{th} coordinate of any point in $\mathbb{V}(I)$. Therefore there are $\leq \prod_{i=1}^n m_i$ possible points in $\mathbb{V}(I)$. ✓

□

5 5.3.10.(c) [See Maple attachment.]

6 6.4.6. Translate the following theorem into a system of polynomial equations: Let ΔABC be a plane triangle, and let M_1, M_2, M_3 be the midpoints of $\overline{BC}, \overline{AC}, \overline{AB}$ respectively. Then $\overline{AM_1}$, $\overline{BM_2}$ and $\overline{CM_3}$ meet in a single point M (the centroid of the triangle).

We can rewrite the conclusion, $\overline{AM_1}$, $\overline{BM_2}$ and $\overline{CM_3}$ meet in a single point M, as follows:
The intersection M of $\overline{AM_1}$ and $\overline{BM_2}$ lies on $\overline{CM_3}$.
This is because it already must be the case that $\overline{AM_1}$ and $\overline{BM_2}$ meet in a single point M ($\overline{AM_1}$ and $\overline{BM_2}$ are the diagonals of the quadrilateral ABM_1M_2). Then $\overline{AM_1}$, $\overline{BM_2}$ and $\overline{CM_3}$ meet in a single point iff $\overline{CM_3}$ intersects both $\overline{AM_1}$ and $\overline{BM_2}$ at M, i.e. iff M lies on $\overline{CM_3}$.



Now WLOG we can fix $A=(0,0)$ and orient and scale the triangle so that $B=(1,0)$. Then $C=(u_1,u_2)$ is arbitrary.
We let $M_1=(x_1,x_2)$, $M_2=(x_3,x_4)$, $M_3=(x_5,x_6)$ and $M=(x_7,x_8)$. These points are determined by A, B and C.

We translate the hypotheses into polynomial equations as follows:

M_1 is the midpoint of $\overline{BC}$

$\Rightarrow B, M_1, C$ are collinear $\Rightarrow \frac{x_2-0}{x_1-1}=\frac{u_2-0}{u_1-1} \Rightarrow x_2(u_1-1)=u_2(x_1-1)$

and $BM_1 = M_1C \Rightarrow (BM_1)^2=(M_1C)^2 \Rightarrow (x_1-1)^2+x_2^2=(u_1-x_1)^2+(u_2-x_2)^2$

M_2 is the midpoint of $\overline{AC}$

$\Rightarrow A, M_2, C$ are collinear $\Rightarrow x_4u_1=u_2x_3$

and $AM_2=M_2C \Rightarrow x_3^2+x_4^2=(u_1-x_3)^2+(u_2-x_4)^2$

M_3 is the midpoint of $\overline{AB}$

$\Rightarrow A, M_3, C$ are collinear $\Rightarrow x_6\cdot 1 = 0\cdot x_5$

and $AM_3=M_3C \Rightarrow x_5^2+x_6^2=(1-x_5)^2+x_6^2$

M is the intersection of $\overline{AM_1}$ and $\overline{BM_2}$

$\Rightarrow A, M, M_1$ are collinear $\Rightarrow x_8x_1=x_2x_7$

B, M, M_2 are collinear $\Rightarrow x_8(x_3-1)=x_4(x_7-1)$.

We translate the conclusion into a polynomial equation as follows:

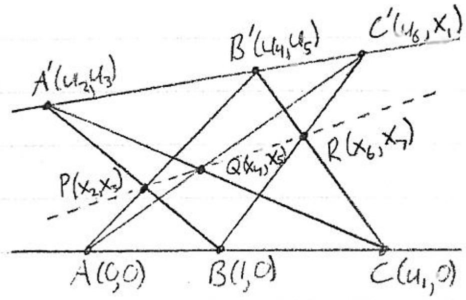
M lies on $\overline{CM_3} \Rightarrow C, M, M_3$ are collinear $\Rightarrow (x_8-x_6)(u_1-x_5) = (u_2-x_6)(x_7-x_5)$.

For the translation of these polynomial equations into polynomials, and to prove that the conclusion polynomial follows algebraically from the hypothesis polynomials, <u>see the Maple attachment.</u>

6 6.4.8. Translate the following theorem of Pappus into a system of polynomial equations: Let A, B, C and A', B', C' be two collinear triples of points, and let

$P = \overline{AB'} \cap \overline{A'B}$, $Q = \overline{AC'} \cap \overline{A'C}$, $R = \overline{BC'} \cap \overline{B'C}$.

Then P, Q, R are collinear points.



WLOG fix $A = (0,0)$ and orient and scale the set of points so that $B = (1,0)$. Let $C = (u_1, 0)$ to satisfy that A, B, C are collinear. Let $A' = (u_2, u_3)$ and $B' = (u_4, u_5)$ be arbitrary, and let $C' = (u_6, x_1)$. Since A', B', C' are collinear, x_1 will be determined by $u_2 - u_6$. Let $P = (x_2, x_3)$, $Q = (x_4, x_5)$ and $R = (x_6, x_7)$. These points are determined by A, B, C, A', B' and C'.

We translate the hypotheses into polynomial equations as follows:

A', B', C' collinear $\Rightarrow \frac{u_5-u_3}{u_4-u_2} = \frac{x_1-u_3}{u_6-u_2} \Rightarrow (u_5-u_3)(u_6-u_2) = (x_1-u_3)(u_4-u_2)$

$P = \overline{AB'} \cap \overline{A'B} \Rightarrow A, P, B'$ and A', P, B are collinear triples

$\Rightarrow x_3 u_4 = u_5 x_2$ and $x_3(u_2-1) = u_3(x_2-1)$

$Q = \overline{AC'} \cap \overline{A'C} \Rightarrow A, Q, C'$ and A', Q, C are collinear triples

$\Rightarrow x_5 u_6 = x_1 x_4$ and $x_5(u_2-u_1) = u_3(x_4-u_1)$

$R = \overline{BC'} \cap \overline{B'C} \Rightarrow B, R, C'$ and B', R, C are collinear triples

$\Rightarrow x_7(u_6-1) = x_1(x_6-1)$ and $x_7(u_4-u_1) = u_5(x_6-u_1)$.

We translate the conclusion into a polynomial equations as follows:

P, Q, R are collinear $\Rightarrow (x_5-x_3)(x_6-x_2) = (x_7-x_3)(x_4-x_2)$.

For the translation of these polynomial equations into polynomials, and to prove that the conclusion polynomial follows algebraically from the hypothesis polynomials, <u>see the Maple attachment.</u>

2 6.4.10. [See Maple attachment.]

10 6.4.11. [See Maple attachment.]

Additional Exercise 1. Let I be an ideal in $k[x_1,\dots,x_n]$ and $R = k[x_1,\dots,x_n]/I$.
Prove that R is an integral domain iff I is prime (over k).

Proof: I is not prime $\Rightarrow \exists$ polys $f, g \notin I$ s.t. $fg \in I$
$\Rightarrow f, g \not\equiv 0 \bmod I$ and $fg \equiv 0 \bmod I$
$\Rightarrow [f], [g] \neq [0] \in R$ and $[f][g] = [fg] = [0]$
$\Rightarrow R$ is not an integral domain. ✓

R is not an integral domain
$\Rightarrow \exists$ classes $[f], [g] \neq [0] \in R$ s.t. $[f][g] = [fg] = [0]$
$\Rightarrow f, g \not\equiv 0 \bmod I$ and $fg \equiv 0 \bmod I$
$\Rightarrow f, g \notin I$ and $fg \in I$
5 $\Rightarrow I$ is not prime. ✓

So R is not an integral domain iff I is not prime.
$\therefore R$ is an integral domain iff I is prime. □

Prove that R is a field iff I is maximal (over k).

Proof: Let $I = \langle f_1,\dots,f_s\rangle \subset k[x_1,\dots,x_n]$.

Suppose that I is maximal.
Let $[a] \in R$, $[a] \neq [0]$. Then $a \notin I$
I is maximal $\Rightarrow \langle I, a\rangle = \langle 1\rangle$ ~~(since I is maximal)~~
$\Rightarrow 1 \in \langle I, a\rangle = \langle f_1,\dots,f_s, a\rangle$
$\Rightarrow \exists\, a_1,\dots,a_s, b \in k[x_1,\dots,x_n]$ s.t. $1 = a_1f_1 + \dots + a_sf_s + ba$
$\Rightarrow [1] = [a_1f_1 + \dots + a_sf_s + ba]$
$= [a_1f_1 + \dots + a_sf_s] + [ab]$
$= [0] + [a][b]$ (since $a_1f_1 + \dots + a_sf_s \in I$)
$= [a][b]$ ✓
$\Rightarrow [b]$ is the multiplicative inverse of $[a] \in R$.
Since any nonzero $[a] \in R$ has an inverse, R is a field

Suppose that R is a field.
Let $a \in k[x_1,\dots,x_n] \setminus I$. Then $[a] \neq [0] \in R$.
$\Rightarrow \exists [b] \in R$ s.t. $[a][b] = [1]$ (since R is a field)
$\Rightarrow [ab] = [1]$
$\Rightarrow 1 - ab \in I$ ✓
$\Rightarrow \exists a_1,\dots,a_s \in k[x_1,\dots,x_n]$ s.t. $1-ab = a_1f_1 + \dots + a_sf_s$ ✓
$\Rightarrow 1 = a_1f_1 + \dots + a_sf_s + ab$ ✓
$\Rightarrow 1 \in \langle I, a\rangle = \langle f_1,\dots,f_s,a\rangle$ for any $a \notin I$
$\Rightarrow \langle I,a\rangle = \langle 1\rangle = k[x_1,\dots,x_n]$.
5 So if J is an ideal strictly containing I then $J = k[x_1,\dots,x_n]$
(since $I \subsetneq J \Rightarrow \exists a \in J \setminus I \Rightarrow \langle I,a\rangle \subset J$, but $\langle I,a\rangle = \langle 1\rangle$
so $\langle 1\rangle \subset J$). Thus I is maximal. $\square$
✓ OK

Additional Exercise 2.

Is $\mathbb{Q}[x,y]/\langle x^2-y^2+1\rangle$ an integral domain?

4 Let $I = \langle x^2-y^2+1\rangle$. Then $I = \mathbb{I}(\mathbb{V}(x^2-y^2+1))$ where $\mathbb{V}(x^2-y^2+1)$ is the hyperbola $y^2-x^2=1$. This hyperbola can be parametrized by
$$x = \frac{2t}{1-t^2},\quad y = \frac{1+t^2}{1-t^2},\quad t \in \mathbb{R}\setminus\{\pm 1\}$$ – work through Exercise 5 in §1.3.

So by Proposition 6 in §4.5, $\mathbb{V}(x^2-y^2+1)$ is irreducible. ✓
Then I is prime (by Proposition 3), so $\mathbb{Q}[x,y]/I$ is an integral domain.

Is $\mathbb{Q}[x,y]/\langle x^2+1, y^2+1\rangle$ an integral domain?

✓ Let $I = \langle x^2+1, y^2+1\rangle$. Then $(x^2+1)-(y^2+1) = x^2-y^2 = (x+y)(x-y) \in I$, but $x+y,\ x-y \notin I$. So I is not prime, thus $\mathbb{Q}[x,y]/I$ is not an integral domain.

4 Additional Exercise 3.
19 Additional Exercise 4 (10 Circles Problem).
[See Maple attachment.]

MATH 800 Assignment 6
(due August 10, 2006, 10:00)

```
> restart;
```

5.3.5

a)

We enter the generators and compute a Groebner basis for $\langle I \rangle$.

```
> F := [y+x^2-1, x*y-2*y^2+2*y];
  G := Groebner[gbasis]( F, plex(x,y) );
```

$$F := [x^2 + y - 1, x\,y - 2\,y^2 + 2\,y]$$

$$G := [4\,y^3 - 7\,y^2 + 3\,y, x\,y - 2\,y^2 + 2\,y, x^2 + y - 1]$$

By Proposition 1, the elements of $R[x, y]/I$ are represented by linear combinations of the monomials not in $\langle \mathrm{LT}(I) \rangle = \langle \mathrm{LT}(G) \rangle$. The monomials in $R[x, y]$ not divisible by y^3, $x\,y$ or x^2 are the 4 listed below.

```
> B := [1,x,y,y^2];
```

$$B := [1, x, y, y^2]$$

3

So the classes of these 4 monomials span $R[x, y]/I$. Also by Proposition 1, these 4 monomials are linearly independent modulo I, that is, the classes of these 4 monomials are linearly independent. Therefore the set $\{[1], [x], [y], [y^2]\}$ forms a basis for $R[x, y]/I$ as a vector space over R, with dimension 4.

It follows that $R[x, y]/I$ is isomorphic to R^4, and we claim that the bijection $\phi : R[x, y]/I \to R^4$ defined by $\phi(a_1\,[1] + a_2\,[x] + a_3\,[y] + a_4\,[y^2]) = [a_1, a_2, a_3, a_4]$ is an isomorphism.

We have $\phi(a_1\,[1] + a_2\,[x] + a_3\,[y] + a_4\,[y^2]) + \phi(b_1\,[1] + b_2\,[x] + b_3\,[y] + b_4\,[y^2]) = [a_1, a_2, a_3, a_4] + [b_1, b_2, b_3, b_4] = [a_1 + b_1, a_2 + b_2, a_3 + b_3, a_4 + b_4] = \phi((a_1 + b_1)\,[1] + (a_2 + b_2)\,[x] + (a_3 + b_3)\,[y] + (a_4 + b_4)\,[y^2]) = \phi(a_1\,[1] + a_2\,[x] + a_3\,[y] + a_4\,[y^2] + b_1\,[1] + b_2\,[x] + b_3\,[y] + b_4\,[y^2])$ and $c\,\phi(a_1\,[1] + a_2\,[x] + a_3\,[y] + a_4\,[y^2]) = c\,[a_1, a_2, a_3, a_4] = [c\,a_1, c\,a_2, c\,a_3, c\,a_4] = \phi(c\,a_1\,[1] + c\,a_2\,[x] + c\,a_3\,[y] + c\,a_4\,[y^2]) = \phi(c\,(a_1\,[1] + a_2\,[x] + a_3\,[y] + a_4\,[y^2]))$. So ϕ preserves vector addition and scalar multiplication, therefore ϕ is a vector space isomorphism.

b)

We compute a multiplication table for $\{[1], [x], [y], [y^2]\}$ in $R[x, y]/I$ (w.r.t. lexicographic order, $y < x$).

2

```
> n := nops(B):
  M := Matrix(n,n):
  for i to n do for j to n do
    M[i,j] := Groebner[normalf]( B[i]*B[j], G, plex(x,y) );
  od od:
```

```
M;
```

$$\begin{bmatrix} 1 & x & y & y^2 \\ x & -y+1 & 2y^2-2y & \frac{3}{2}y^2-\frac{3}{2}y \\ y & 2y^2-2y & y^2 & \frac{7}{4}y^2-\frac{3}{4}y \\ y^2 & \frac{3}{2}y^2-\frac{3}{2}y & \frac{7}{4}y^2-\frac{3}{4}y & \frac{37}{16}y^2-\frac{21}{16}y \end{bmatrix}$$

c)

The quotient ring $R[x, y]/I$ is not a field. In fact, it is not even an integral domain. Consider $[x]$ and $\left[y^2-\frac{3y}{4}\right]$ in $R[x, y]/I$. Neither of these equal $[0]$ yet their product is $[0]$, so they are zero divisors in $R[x, y]/I$. ✓

2

```
> [Groebner[normalf]( x, G, plex(x,y) )]*[Groebner[normalf](
  y^2-3/4*y, G, plex(x,y) )] = [Groebner[normalf](
  x*(y^2-3/4*y), G, plex(x,y) )];
```

$$[x]\left[y^2-\frac{3}{4}y\right]=[0]$$

d)

We now want to compute the inverse of the class of $f = 1 + y$ in $R[x, y]/I$. We know that if this inverse exists, it must be represented by a polynomial of the form $a_1 + a_2 x + a_3 y + a_4 y^2$ and must satisfy the following equation.

```
> f := 1+y:
  f_inv := add( a[i]*B[i], i=1..n ):
  f*f_inv = 1;
```

$$(1+y)(a_1+a_2x+a_3y+a_4y^2)=1$$

We reduce the above equation modulo I, equate the coefficients and solve for the a_i. We then check that we have indeed found an inverse of $[f]$.

2

```
> Groebner[normalf]( f*f_inv, G, plex(x,y) );
  coeffs( %, [x,y], 'm' ):
  {seq( %[i] = `if`( m[i] = 1, 1, 0 ), i=1..n )};
  solve( %, {seq( a[i], i=1..n )} );
  subs( %, f*f_inv = Groebner[normalf]( f*f_inv, G, plex(x,y) )
  );
```

$$a_1+\left(\frac{11}{4}a_4+a_3+2a_2\right)y^2+\left(a_3+a_1-2a_2-\frac{3}{4}a_4\right)y+a_2x$$

$$\left\{a_2=0,\ a_1=1,\ \frac{11}{4}a_4+a_3+2a_2=0,\ a_3+a_1-2a_2-\frac{3}{4}a_4=0\right\}$$

$$\{a_2 = 0, a_1 = 1, a_4 = \frac{2}{7}, a_3 = \frac{-11}{14}\}$$

$$(1+y)\left(1 - \frac{11}{14}y + \frac{2}{7}y^2\right) = 1$$

So the class of $1 - \frac{11\,y}{14} + \frac{2\,y^2}{7}$ is an inverse of the class of $1+y$ in $R\,[x, y]/I$. Since we are in a commutative ring, we know that inverses must be unique. So we have found *the* inverse of the class of $1+y$.

5

5.3.10

We compute a Groebner basis for the polynomials for the algebraic extensions $\sqrt{2}, \sqrt{3}$.

```
> G := Groebner[gbasis]( [y^2-2, z^2-3], plex(y,z) );
```

$$G := [z^2 - 3, y^2 - 2]$$

The possible monomials in the remainder representatives of $Q\,[y, z]/I$, $I = \langle G \rangle$, are the following.

```
> B := [1,z,y,y*z];
```

$$B := [1, z, y, y\,z]$$

We now want to compute the inverse of $f = x + y + z$ in $(Q\,[y, z]/I)(x)$, of the form $a_1 + a_2\,z + a_3\,y + a_4\,y\,z$ and satisfying the following equation.

```
> f := x+y+z:
  f_inv := add( a[i]*B[i], i=1..n ):
  f*f_inv = 1;
```

$$(x+y+z)\,(a_1 + a_2\,z + a_3\,y + a_4\,y\,z) = 1$$

We reduce the above equation modulo I, equate the coefficients and solve for the a_i. We check that this gives the inverse.

```
> Groebner[normalf]( f*f_inv, G, plex(y,z) );
  coeffs( %, [y,z], 'm' ):
  {seq( %[i] = `if`( m[i] = 1, 1, 0 ), i=1..n )};
  cf := solve( %, {seq( a[i], i=1..n )} );
  subs( cf, f*f_inv = Groebner[normalf]( f*f_inv, G, plex(y,z) )
  ):
  simplify(%);
```

$$x\,a_1 + 2\,a_3 + 3\,a_2 + (x\,a_3 + a_1 + 3\,a_4)\,y + (a_2\,x + a_1 + 2\,a_4)\,z + (x\,a_4 + a_2 + a_3)\,y\,z$$

$$\{x\,a_4 + a_2 + a_3 = 0, x\,a_3 + a_1 + 3\,a_4 = 0, a_2\,x + a_1 + 2\,a_4 = 0, x\,a_1 + 2\,a_3 + 3\,a_2 = 1\}$$

$$cf := \{a_4 = \frac{2\,x}{x^4 - 10\,x^2 + 1}, a_3 = -\frac{x^2 + 1}{x^4 - 10\,x^2 + 1}, a_2 = -\frac{x^2 - 1}{x^4 - 10\,x^2 + 1}, a_1 = \frac{x\,(x^2 - 5)}{x^4 - 10\,x^2 + 1}\}$$

$$-\frac{(x+y+z)\,(-x^3 + 5\,x + z\,x^2 - z + x^2\,y + y - 2\,x\,y\,z)}{x^4 - 10\,x^2 + 1} = 1$$

Now, letting $y = \sqrt{2}, z = \sqrt{3}$, we get that $(x + \sqrt{2} + \sqrt{3})\,(a_1 + a_2\sqrt{3} + a_3\sqrt{2} + a_4\sqrt{2}\,\sqrt{3}) = 1$.

This means that $r = \dfrac{1}{x+\sqrt{2}+\sqrt{3}}$ equals $a_1 + a_2\sqrt{3} + a_3\sqrt{2} + a_4\sqrt{2}\sqrt{3}$, the following expression.

```
> r := simplify( subs( {y=sqrt(2), z=sqrt(3)} union cf, f_inv )
  );
```

$$r := \frac{x^3 - 5x - \sqrt{3}\,x^2 + \sqrt{3} - x^2\sqrt{2} - \sqrt{2} + 2x\sqrt{2}\sqrt{3}}{x^4 - 10x^2 + 1}$$

We check that this is the case.

```
> evalb( simplify(expand( r*(x+sqrt(2)+sqrt(3)) )) = 1 );
```

$$true$$

Scott Cowan
200034814
scottc@sfu.ca

```
>
```

MATH 800 Assignment 6

(due August 10, 2006, 10:00)

```
> restart;
```

6.4.6

We enter the equations for the hypotheses and the conclusion of the theorem.

```
> hyp := [x[2]*(u[1]-1) = u[2]*(x[1]-1), (x[1]-1)^2+x[2]^2 =
  (u[1]-x[1])^2+(u[2]-x[2])^2, x[4]*u[1] = u[2]*x[3],
  x[3]^2+x[4]^2 = (u[1]-x[3])^2+(u[2]-x[4])^2, x[6]*1 = 0*x[5],
  x[5]^2+x[6]^2 = (1-x[5])^2+x[6]^2, x[8]*x[1] = x[2]*x[7],
  x[8]*(x[3]-1) = x[4]*(x[7]-1)];
  concl := (x[8]-x[6])*(u[1]-x[5]) = (u[2]-x[6])*(x[7]-x[5]);
```

$$hyp := [x_2(u_1-1) = u_2(x_1-1), (x_1-1)^2+x_2^2 = (u_1-x_1)^2+(u_2-x_2)^2, x_4 u_1 = u_2 x_3, x_3^2+x_4^2 = (u_1-x_3)^2+(u_2-x_4)^2, x_6 = 0, x_5^2+x_6^2 = (1-x_5)^2+x_6^2, x_8 x_1 = x_2 x_7, x_8(x_3-1) = x_4(x_7-1)]$$

$$concl := (x_8-x_6)(u_1-x_5) = (u_2-x_6)(x_7-x_5)$$

We convert the equations into polynomials.

```
> H := [seq( lhs(e)-rhs(e), e=hyp )];
  g := lhs(concl)-rhs(concl);
```

$$H := [x_2(u_1-1)-u_2(x_1-1), (x_1-1)^2+x_2^2-(u_1-x_1)^2-(u_2-x_2)^2, x_4 u_1-u_2 x_3, x_3^2+x_4^2-(u_1-x_3)^2-(u_2-x_4)^2, x_6, x_5^2-(1-x_5)^2, x_8 x_1-x_2 x_7, x_8(x_3-1)-x_4(x_7-1)]$$

$$g := (x_8-x_6)(u_1-x_5)-(u_2-x_6)(x_7-x_5)$$

To verify that the conclusion follows from the hypotheses, we compute a Groebner basis for the hypothesis polynomials (with respect to the variables $x_1, \ldots, x_8$), then we divide the conclusion polynomial by this Groebner basis.

```
> V := [seq( x[i], i=1..8 )];
  G := Groebner[gbasis]( H, tdeg(op(V)) );
  Groebner[normalf]( g, G, tdeg(op(V)) );
```

$$V := [x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8]$$

$$G := [3x_8-u_2, 3x_7-1-u_1, x_6, -1+2x_5, 2x_4-u_2, -u_1+2x_3, 2x_2-u_2, -u_1+2x_1-1]$$

$$0$$

The conclusion polynomial reduces to zero, proving (by Proposition 5) that the conclusion follows from the hypotheses.

6.4.8

We enter the equations for the hypotheses and the conclusion of the theorem.

```
> hyp := [(u[5]-u[3])*(u[6]-u[2]) = (x[1]-u[3])*(u[4]-u[2]),
  x[3]*u[4] = u[5]*x[2], x[3]*(u[2]-1) = u[3]*(x[2]-1), x[5]*u[6]
```

```
   = x[1]*x[4], x[5]*(u[2]-u[1]) = u[3]*(x[4]-u[1]), x[7]*(u[6]-1)
   = x[1]*(x[6]-1), x[7]*(u[4]-u[1]) = u[5]*(x[6]-u[1])];
   concl := (x[5]-x[3])*(x[6]-x[2]) = (x[7]-x[3])*(x[4]-x[2]);
```

$$hyp := [(u_5 - u_3)(u_6 - u_2) = (x_1 - u_3)(u_4 - u_2), x_3 u_4 = u_5 x_2, x_3 (u_2 - 1) = u_3 (x_2 - 1), x_5 u_6 = x_4 x_1, x_5 (u_2 - u_1) = u_3 (x_4 - u_1), x_7 (u_6 - 1) = x_1 (x_6 - 1), x_7 (u_4 - u_1) = u_5 (x_6 - u_1)]$$

$$concl := (x_5 - x_3)(x_6 - x_2) = (x_7 - x_3)(x_4 - x_2)$$

We convert the equations into polynomials.

```
> H := [seq( lhs(e)-rhs(e), e=hyp )];
  g := lhs(concl)-rhs(concl);
```

$$H := [(u_5 - u_3)(u_6 - u_2) - (x_1 - u_3)(u_4 - u_2), x_3 u_4 - u_5 x_2, x_3 (u_2 - 1) - u_3 (x_2 - 1), x_5 u_6 - x_4 x_1, x_5 (u_2 - u_1) - u_3 (x_4 - u_1), x_7 (u_6 - 1) - x_1 (x_6 - 1), x_7 (u_4 - u_1) - u_5 (x_6 - u_1)]$$

$$g := (x_5 - x_3)(x_6 - x_2) - (x_7 - x_3)(x_4 - x_2)$$

To verify that the conclusion follows from the hypotheses, we compute a Groebner basis for the hypothesis polynomials (with respect to the variables $x_1, \ldots, x_7$), then we divide the conclusion polynomial by this Groebner basis.

```
> V := [seq( x[i], i=1..8 )];
  G := Groebner[gbasis]( H, tdeg(op(V)) );
  Groebner[normalf]( g, G, tdeg(op(V)) );
```

$$V := [x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8]$$

$$\begin{aligned} G := [&-u_5 u_3 u_6 + u_5 x_7 u_2 - u_5 x_7 u_4 - u_3 u_4^2 x_7 + u_5^2 u_2 u_1 - u_5^2 u_6 u_1 + u_5 u_3 u_4 - u_5 x_7 u_2 u_6 \\ &+ u_5 u_2 x_7 u_4 + u_5 u_6 x_7 u_1 - u_5 u_2 x_7 u_1 + u_3 u_6 x_7 u_4 - u_3 u_6 x_7 u_1 + u_3 u_6 u_5 u_1 + u_3 u_4 x_7 u_1 \\ &- u_3 u_4 u_5 u_1 + u_5^2 u_6 - u_5^2 u_2, -u_5 x_6 u_2 u_1 - u_5 x_6 u_4 + u_5 u_4 u_1 - u_5 x_6 u_2 u_6 + x_6 u_5 u_2 \\ &- u_4 u_5 u_6 u_1 - u_5 u_2 u_4 - u_5 u_6 u_1 + u_5 u_2 u_6 u_1 + u_5 x_6 u_2 u_4 + u_5 x_6 u_6 u_1 + u_4 u_5 u_6 + u_3 u_6 u_1 \\ &- u_3 u_4 u_1 - x_6 u_3 u_4^2 + x_6 u_3 u_6 u_4 - x_6 u_3 u_6 u_1 + x_6 u_3 u_4 u_1 + u_3 u_4^2 - u_3 u_6 u_4, u_4 u_3 x_5 u_6 \\ &- u_2 x_5 u_5 u_6 + u_2^2 x_5 u_5 - u_2 x_5 u_3 u_4 + x_5 u_1 u_5 u_6 - x_5 u_1 u_5 u_2 - x_5 u_1 u_3 u_6 + x_5 u_1 u_3 u_4 \\ &- u_3 u_6 u_5 u_1 + u_3 u_1 u_5 u_2 + u_3^2 u_1 u_6 - u_3^2 u_1 u_4, -u_3 x_4 u_2 u_4 + u_3 x_4 u_6 u_4 - u_3 x_4 u_6 u_1 \\ &+ u_3 x_4 u_4 u_1 + u_3 u_2 u_1 u_6 - u_3 u_6 u_4 u_1 + x_4 u_5 u_6 u_1 - x_4 u_5 u_2 u_6 + x_4 u_2^2 u_5 - x_4 u_5 u_2 u_1, \\ &x_3 u_5 u_2 - x_3 u_5 - u_3 x_3 u_4 + u_5 u_3, u_5 u_2 x_2 - u_5 x_2 - x_2 u_3 u_4 + u_3 u_4, \\ &u_5 u_6 - u_5 u_2 - u_3 u_6 - u_4 x_1 + u_2 x_1 + u_3 u_4] \end{aligned}$$

$$0$$

The conclusion polynomial reduces to zero, proving (by Proposition 5) that the conclusion follows from the hypotheses.

6.4.10

We enter the hypothesis and conclusion polynomials.

```
> h1  := x[2]-u[3];
  h2  := (x[1]-u[1])*u[3]-u[2]*x[2];
  h3  := x[4]*x[1]-x[3]*u[3];
  h4  := x[4]*(u[2]-u[1])-(x[3]-u[1])*u[3];
  g  := x[1]^2-2*x[1]*x[3]-2*x[4]*x[2]+x[2]^2;
```

$$h1 := x_2 - u_3$$

$$h2 := (x_1 - u_1)\, u_3 - u_2\, x_2$$

$$h3 := x_4\, x_1 - x_3\, u_3$$

$$h4 := x_4\,(u_2 - u_1) - (x_3 - u_1)\, u_3$$

$$g := x_1^2 - 2\, x_1\, x_3 - 2\, x_4\, x_2 + x_2^2$$

We compute the reduced Groebner basis for $\langle h_1, h_2, h_3, h_4, 1 - y\, g\rangle$ to see that it is not $\{1\}$.

```
> Groebner[gbasis]( [h1,h2,h3,h4,1-y*g],
  tdeg(u[1],u[2],u[3],x[1],x[2],x[3],x[4],y) );
```

$$[-x_2 + u_3,\ x_4\, u_2 - x_4\, x_1,\ x_4\, u_1,\ x_3\, x_2 - x_4\, x_1,\ u_2\, x_2 - x_1\, x_2,\ u_1\, x_2,$$
$$-1 + y\, x_1^2 - 2\, y\, x_1\, x_3 - 2\, y\, x_4\, x_2 + y\, x_2^2]$$

6.4.11

a)

We compute the lexicographic Groebner basis for $\langle h_1, h_2, h_3, h_4\rangle$.

```
> V := x[1],x[2],x[3],x[4],u[1],u[2],u[3]:
  G := Groebner[gbasis]( [h1,h2,h3,h4], plex(V) );
```

$$G := [2\, x_4\, u_1\, u_3 - u_1\, u_3^2,\ 2\, x_4\, u_1^2 - 2\, x_4\, u_1\, u_2 - u_1^2\, u_3 + u_1\, u_2\, u_3,$$
$$x_3\, u_3 + x_4\, u_1 - x_4\, u_2 - u_1\, u_3,\ x_2 - u_3,\ x_1\, u_3 - u_1\, u_3 - u_2\, u_3,\ x_1\, x_4 + x_4\, u_1 - x_4\, u_2 - u_1\, u_3]$$

We reorder and label the polynomials in this basis to match the example in the text.

```
> for i to nops(G) do f||i := G[-i]/lcoeff( G[-i], [V] ) od;
```

$$f1 := x_1\, x_4 + x_4\, u_1 - x_4\, u_2 - u_1\, u_3$$

$$f2 := x_1\, u_3 - u_1\, u_3 - u_2\, u_3$$

$$f3 := x_2 - u_3$$

$$f4 := x_3\, u_3 + x_4\, u_1 - x_4\, u_2 - u_1\, u_3$$

$$f5 := x_4\, u_1^2 - x_4\, u_1\, u_2 - \frac{1}{2}\, u_1^2\, u_3 + \frac{1}{2}\, u_1\, u_2\, u_3$$

$$f6 := x_4\, u_1\, u_3 - \frac{1}{2}\, u_1\, u_3^2$$

Now f_2 factors:

```
> factor(f2);
```

$$u_3\,(x_1 - u_1 - u_2)$$

so $\mathrm{V}(f_1, f_2, f_3, f_4, f_5, f_6) = \mathrm{V}(f_1, f_3, f_4, f_5, f_6) \cap \mathrm{V}(f_2) =$
$\mathrm{V}(f_1, f_3, f_4, f_5, f_6) \cap (\mathrm{V}(x_1 - u_1 - u_2) \cup \mathrm{V}(u_3))$ by Lemma 2 of section 1.2
$= (\mathrm{V}(f_1, f_3, f_4, f_5, f_6) \cap \mathrm{V}(x_1 - u_1 - u_2)) \cup (\mathrm{V}(f_1, f_3, f_4, f_5, f_6) \cap \mathrm{V}(u_3))$
$= \mathrm{V}(f_1, f_3, f_4, f_5, f_6, x_1 - u_1 - u_2) \cup \mathrm{V}(f_1, f_3, f_4, f_5, f_6, u_3)$ by Lemma 2.
Alternatively we can check that
$\langle f_1, x_1 - u_1 - u_2, f_3, f_4, f_5, f_6 \rangle \cap \langle f_1, u_3, f_3, f_4, f_5, f_6 \rangle = \langle f_1, f_2, f_3, f_4, f_5, f_6 \rangle$. Then
$\mathrm{V}(\langle f_1, x_1 - u_1 - u_2, f_3, f_4, f_5, f_6 \rangle) \cup \mathrm{V}(\langle f_1, u_3, f_3, f_4, f_5, f_6 \rangle) =$
$\mathrm{V}(\langle f_1, x_1 - u_1 - u_2, f_3, f_4, f_5, f_6 \rangle \cap \langle f_1, u_3, f_3, f_4, f_5, f_6 \rangle) = \mathrm{V}(\langle f_1, f_2, f_3, f_4, f_5, f_6 \rangle)$.

```
> Gt := Groebner[gbasis]( [op(expand(
  t*[f1,x[1]-u[1]-u[2],f3,f4,f5,f6] )), op(expand(
  (1-t)*[f1,u[3],f3,f4,f5,f6] ))], plex(t,V) ):
  remove( has, Gt, t );
  evalb( % = G );
```

2

$$[2x_4u_1u_3 - u_1u_3^2,\ 2x_4u_1^2 - 2x_4u_1u_2 - u_1^2u_3 + u_1u_2u_3,\ x_3u_3 + x_4u_1 - x_4u_2 - u_1u_3,$$
$$x_2 - u_3,\ x_1u_3 - u_1u_3 - u_2u_3,\ x_1x_4 + x_4u_1 - x_4u_2 - u_1u_3]$$

$$true$$

b, c)

To further decompose these varieties, we compute Groebner bases for $\langle f_1, x_1 - u_1 - u_2, f_3, f_4, f_5, f_6 \rangle$, $\langle f_1, u_3, f_3, f_4, f_5, f_6 \rangle$ and factor them.

3

```
> Groebner[gbasis]( [op(G), x[1]-u[1]-u[2]], plex(V) );
  factor(G1);
  G2 := Groebner[gbasis]( [op(G), u[3]], plex(V) );
  factor(G2);
```

$$[2x_4u_1 - u_1u_3,\ 2x_3u_3 - 2x_4u_2 - u_1u_3,\ x_2 - u_3,\ x_1 - u_1 - u_2]$$
$$[-u_1(-2x_4 + u_3),\ 2x_3u_3 - 2x_4u_2 - u_1u_3,\ x_2 - u_3,\ x_1 - u_1 - u_2]$$
$$G2 := [u_3,\ x_4u_1 - x_4u_2,\ x_2,\ x_1x_4]$$
$$[u_3,\ x_4(u_1 - u_2),\ x_2,\ x_1x_4]$$

The possible solutions from $f_1, f_2, f_3, f_4, f_5, f_6$ are as follows:
- $u_3 = 0$ and $x_4 = 0$
- $u_3 = 0$ and $x_4 \neq 0$ so $u_1 = u_2$
- $u_3 \neq 0$ so $x_1 = u_1 + u_2$, and $u_1 = 0$
- $u_3 \neq 0$ so $x_1 = u_1 + u_2$, and $u_1 \neq 0$ so $x_4 = \dfrac{u_3}{2}$.

These correspond to the varieties U_1, U_2, U_3, V' respectively.

```
> Groebner[gbasis]( [op(G), x[1]-u[1]-u[2], 1-y*u[1]*u[3],
```

```
x[4]-u[3]/2], plex(y,V) ):
Vp := remove( has, %, y );
U1 := Groebner[gbasis]( [op(G), u[3], x[4]], plex(V) );
Groebner[gbasis]( [op(G), u[3], u[1]-u[2], 1-y*x[4]],
plex(y,V) ):
U2 := remove( has, %, y );
Groebner[gbasis]( [op(G), x[1]-u[1]-u[2], 1-y*u[3], u[1]],
plex(y,V) ):
U3 := remove( has, %, y );
```

$$Vp := [2\,x_4 - u_3, 2\,x_3 - u_1 - u_2, x_2 - u_3, x_1 - u_1 - u_2]$$

$$U1 := [u_3, x_4, x_2]$$

$$U2 := [u_3, u_1 - u_2, x_2, x_1]$$

$$U3 := [u_1, x_3\,u_3 - x_4\,u_2, x_2 - u_3, x_1 - u_2]$$

d)

Now V' can be parametrized by $u_1 = t_1$, $u_2 = t_2$, $u_3 = t_3$, $x_1 = t_1 + t_2$, $x_2 = t_3$, $x_3 = \frac{t_1 + t_2}{2}$, $x_4 = \frac{t_3}{2}$; U_1 can be parametrized by $x_2, x_4, u_3 = 0$ and $x_1 = t_1$, $x_3 = t_2$, $u_1 = t_3$, $u_2 = t_4$; U_2 can be parametrized by $x_1, x_2, u_3 = 0$ and $x_3 = t_1$, $x_4 = t_2$, $u_1 = t_3$, $u_2 = t_3$; U_3 can be parametrized by $u_1 = 0$ and $u_2 = t_1$, $u_3 = t_2$, $x_1 = t_1$, $x_2 = t_2$, $x_3 = t_3$, $x_4 = \frac{t_2\,t_3}{t_1}$. Then by Propositions 5 & 6 of section 4.5, these 4 varieties are irreducible. So $V = (V' \cup U_1) \cup (U_2 \cup U_3)$ is a decomposition into irreducible varieties.

```
> Groebner[normalf]( Vp[2], U1, plex(V) ), Groebner[normalf](
  Vp[2], U2, plex(V) ), Groebner[normalf]( Vp[2], U3, plex(V)
  );
  Groebner[normalf]( U1[2], Vp, plex(V) ), Groebner[normalf](
  U1[2], U2, plex(V) ), Groebner[normalf]( U1[2], U3, plex(V)
  );
  Groebner[normalf]( U2[2], Vp, plex(V) ), Groebner[normalf](
  U2[2], U1, plex(V) ), Groebner[normalf]( U2[2], U3, plex(V)
  );
  Groebner[normalf]( U3[1], Vp, plex(V) ), Groebner[normalf](
  U3[1], U1, plex(V) ), Groebner[normalf]( U3[1], U2, plex(V)
  );
```

$$2\,x_3 - u_1 - u_2, 2\,x_3 - 2\,u_2, 2\,x_3 - u_2$$

$$\frac{1}{2}\,u_3, x_4, x_4$$

$$u_1 - u_2, u_1 - u_2, -u_2$$

$$u_1, u_1, u_2$$

2 The previous calculation shows that each of the ideals corresponding to V', U_1, U_2, U_3 contains a polynomial not in any of the others. It follows here that none of these varieties is contained in any of the others. Therefore we have found a minimal (irredundant) decomposition for V.

e)

1 The conclusion of the theorem is valid on the component V', as shown by the following calculation.

```
> Groebner[normalf]( g, Vp, plex(V) );
```

$$0$$

f)

If we work with the alternate polynomials $h_1{}'$, $h_2{}'$ We get the following Groebner basis.

```
> h1  := x[1]-u[1]-u[2];
  h2  := x[2]-u[3];
  G := Groebner[gbasis]( [h1,h2,h3,h4], plex(V) );
  factor(G);
```

$$h1 := x_1 - u_1 - u_2$$

$$h2 := x_2 - u_3$$

$$G := [2\,x_4\,u_1 - u_1\,u_3,\ 2\,x_3\,u_3 - 2\,x_4\,u_2 - u_1\,u_3,\ x_2 - u_3,\ x_1 - u_1 - u_2]$$

$$[-u_1\,(-2\,x_4 + u_3),\ 2\,x_3\,u_3 - 2\,x_4\,u_2 - u_1\,u_3,\ x_2 - u_3,\ x_1 - u_1 - u_2]$$

The first polynomial in the basis factors, so the corresponding variety W = V($h_1{}'$, $h_2{}'$, h_3, h_4) is reducible into at least two components. We compute these.

```
> Groebner[gbasis]( [op(G), u[1]], plex(V) );
  factor(%);
  Groebner[gbasis]( [op(G), u[3]-2*x[4]], plex(V) );
  factor(%);
```

$$[u_1,\ x_3\,u_3 - x_4\,u_2,\ x_2 - u_3,\ x_1 - u_2]$$

$$[u_1,\ x_3\,u_3 - x_4\,u_2,\ x_2 - u_3,\ x_1 - u_2]$$

2

$$[2\,x_4 - u_3,\ 2\,x_3\,u_3 - u_1\,u_3 - u_2\,u_3,\ x_2 - u_3,\ x_1 - u_1 - u_2]$$

$$[2\,x_4 - u_3,\ -u_3\,(-2\,x_3 + u_1 + u_2),\ x_2 - u_3,\ x_1 - u_1 - u_2]$$

The second component is also reducible. We decompose it.

```
> Groebner[gbasis]( [op(G), u[3]-2*x[4], u[3], 1-y*u[1]],
  plex(y,V) ):
  remove( has, %, y );
  Groebner[gbasis]( [op(G), u[3]-2*x[4], 2*u[3]-u[1]-u[2],
  1-y*u[1]*u[3]], plex(y,V) ):
  remove( has, %, y );
```

$$[u_3, x_4, x_2, x_1 - u_1 - u_2]$$

$$[-2\,u_3 + u_1 + u_2, 2\,x_4 - u_3, -u_3 + x_3, x_2 - u_3, x_1 - 2\,u_3]$$

So here W is the union of 3 irreducible components.

Scott Cowan
200034814
scottc@sfu.ca

>

MATH 800 Assignment 6
(due August 10, 2006, 10:00)

```
> restart;
```

Additional Exercise 3

4

We enter the generators for I and compute Groebner bases w.r.t. plex and tdeg orderings.

```
> F := [x^2+y+z-1, x+y^2+z-1, x+y+z^2-1];
  Gp := Groebner[gbasis]( F, plex(x,y,z) );
  Gt := Groebner[gbasis]( F, tdeg(x,y,z) );
```

$$F := [x^2+y+z-1, x+y^2+z-1, x+y+z^2-1]$$

$$Gp := [z^6-4z^4+4z^3-z^2, 2yz^2+z^4-z^2, y^2-y-z^2+z, x+y+z^2-1]$$

$$Gt := [x+y+z^2-1, x+y^2+z-1, x^2+y+z-1]$$

The monomial basis for $C[x,y,z]/I$ w.r.t. a certain monomial ordering is the set of monomials not in $\langle \mathrm{LT}(I)\rangle = \langle \mathrm{LT}(G)\rangle$, where G is a Groebner basis w.r.t. that ordering. The monomial bases w.r.t. plex and tdeg orderings consist of all monomials not divisible by any of $\{z^6, yz^2, y^2, x\}$ or $\{z^2, y^2, x^2\}$, respectively:

```
> Bp := {seq( z^i, i=0..5 ), y, yz};
  Bt := {seq( seq( seq( x^i*y^j*z^k, k=0..1 ), j=0..1 ), i=0..1
  )};
```

$$Bp := \{1, z, y, z^2, z^3, z^4, z^5, yz\}$$

$$Bt := \{1, x, z, y, xz, zy, xy, xyz\}$$

These bases both contain 8 monomials, so $\dim(R) = 8$ where $R = C[x,y,z]/I$. By Proposition 8 of section 5.3, there are at most 8 points in $V(I)$. Since there are actually 5 points in $V(I)$, this is correct.

Now we need to compute $J = \sqrt{I}$. We begin by factoring the first polynomial in the plex Groebner basis for I.

```
> factor(Gp[1]);
```

$$z^2(z^2+2z-1)(z-1)^2$$

$\sqrt{I}$ must contain the square-free part of this polynomial, which we compute.

```
> g := simplify( Gp[1]/gcd( Gp[1], diff(Gp[1],z) ) );
```

$$g := (z^3+z^2-3z+1)z$$

We add this to the generators for I and compute a new Groebner basis.

```
> Grp := Groebner[gbasis]( [op(F),g], plex(x,y,z) );
  factor(Grp);
```

$$Grp := [z^4+z^3-3z^2+z, 2zy+z^3-z, y^2-y-z^2+z, x+y+z^2-1]$$

$$[z(z-1)(z^2+2z-1), z(2y+z^2-1), -(y+z-1)(z-y), x+y+z^2-1]$$

The ideal generated by this basis is indeed radical, so this is a Groebner basis for $J = \sqrt{I}$ w.r.t. plex ordering. We also compute the Groebner basis w.r.t. tdeg ordering.

Proof? Why? Basis factored into distinct linear factors?

```
> Grt := Groebner[gbasis]( Grp, tdeg(x,y,z) );
  factor(Grt);
```

$$Grt := [x+y+z^2-1, x\,z-z\,y, x+y^2+z-1, x\,y-z\,y, x^2+y+z-1]$$

$$[x+y+z^2-1, z\,(x-y), x+y^2+z-1, y\,(x-z), x^2+y+z-1]$$

As before, the monomial basis for $C\,[x, y, z]/J$ w.r.t. a certain monomial ordering is the set of monomials not in $\langle \mathrm{LT}(J) \rangle = \langle \mathrm{LT}(G) \rangle$, where G is a Groebner basis w.r.t. that ordering. The monomial bases w.r.t. plex and tdeg orderings consist of all monomials not divisible by any of $\{z^4, y\,z, y^2, x\}$ or $\{z^2, y^2, x^2, x\,y, x\,z\}$, respectively:

```
> Brp := {seq( z^i, i=0..3 ), y};
  Brt := {seq( seq( y^j*z^k, k=0..1 ), j=0..1 ), x};
```

$$Brp := \{1, z, y, z^2, z^3\}$$

$$Brt := \{1, x, z, y, z\,y\}$$

These bases both contain 5 monomials, so $\dim(R) = 5$ where $R = C\,[x, y, z]/J$. By Proposition 8 of section 5.3, there are exactly 5 points in $\mathrm{V}(J) = \mathrm{V}(I)$, which is correct.

Scott Cowan
200034814
scottc@sfu.ca

```
>
```

MATH 800 Assignment 6

(due August 10, 2006, 10:00)

```
> restart;
```

We load in our code for the FGLM algorithm from the course project. We will use this to compute GBs w.r.t. lexicographic order from GBs w.r.t. total degree (grevlex) order whenever Maple is too slow in computing GBs w.r.t. elimination order.

We also create a procedure that will save us most of the typing in setting up distance equations.

```
> d2 := proc( pt1, pt2 ) return (x[pt2]-x[pt1])^2+(y[pt2]-y[pt1])^2
  end:
  d2(1,2) = r^2;
```

$$(x_2 - x_1)^2 + (y_2 - y_1)^2 = r^2$$

The 10 Circles Problem

Note that the calculations for each arrangment culminate in a labelled plot of the 10 circles. This provides the circle numbering scheme referred to in each part.

a) Show me your labelling.

7

We input the boundary, symmetry and distance equations for the first arrangment. Boundary equations specify which coordinates are touching the outside of the square. In this arrangement, symmetry is found over the diagonal line through [0, 0] and [1, 1] (the line $y = x$) between circle pairs (2, 4) and (7, 9), and over the diagonal line through [1, 0] and [0, 1] (the line $y = 1 - x$) between circle pairs (2, 7) and (4, 9). Distance equations (with distance m) are specified for every pair of touching circles. When the circles are separated by a line parallel to one of the axes we give a simplified linear constraint rather than the general quadratic constraint.

Free circles: In the arrangement, circles 3 and 8 are free. We fix them to lie in the corners [1, 0] and [0, 1] respectively so as to provide a unique solution for the position of every circle.

```
> bdry := [x[1] = 0, y[1] = 0, y[2] = 0, x[3] = 1, y[3] = 0,
  x[4] = 0, x[7] = 1, x[8] = 0, y[8] = 1, y[9] = 1, x[10] = 1,
  y[10] = 1];
  symm := [x[2] = y[4], y[7] = x[9], x[2] = 1-y[7], y[4] =
  1-x[9]];
  dist := [x[2] = x[1]+m, y[4] = y[1]+m, d2(2,6) = m^2, d2(4,5)
  = m^2, d2(5,6) = m^2, d2(5,9) = m^2, d2(6,7) = m^2, y[10] =
  y[7]+m, x[10] = x[9]+m];
```

$bdry :=$

$$[x_1 = 0, y_1 = 0, y_2 = 0, x_3 = 1, y_3 = 0, x_4 = 0, x_7 = 1, x_8 = 0, y_8 = 1, y_9 = 1, x_{10} = 1, y_{10} = 1]$$

$$symm := [x_2 = y_4, y_7 = x_9, x_2 = 1 - y_7, y_4 = 1 - x_9]$$

$dist := [x_2 = x_1 + m, y_4 = y_1 + m, (x_6 - x_2)^2 + (y_6 - y_2)^2 = m^2, (x_5 - x_4)^2 + (y_5 - y_4)^2 = m^2,$

$(x_6 - x_5)^2 + (y_6 - y_5)^2 = m^2, (x_9 - x_5)^2 + (y_9 - y_5)^2 = m^2, (x_7 - x_6)^2 + (y_7 - y_6)^2 = m^2,$

$y_{10} = y_7 + m, x_{10} = x_9 + m]$

We construct a polynomial basis from the given equations, along with a list of the appropriate variables.

```
> F := [seq( subs( bdry, lhs(e)-rhs(e) ), e=[op(symm),op(dist)]
  )];
  V := map( z -> `if`( has(F,z), z, NULL ), [seq(
  op([x[i],y[i]]), i=1..10 )] );
```

$$F := [x_2 - y_4, y_7 - x_9, x_2 - 1 + y_7, y_4 - 1 + x_9, x_2 - m, y_4 - m, (x_6 - x_2)^2 + y_6^2 - m^2, x_5^2 + (y_5 - y_4)^2 - m^2, (x_6 - x_5)^2 + (y_6 - y_5)^2 - m^2, (x_9 - x_5)^2 + (1 - y_5)^2 - m^2, (1 - x_6)^2 + (y_7 - y_6)^2 - m^2, 1 - y_7 - m, 1 - x_9 - m]$$

$$V := [x_2, y_4, x_5, y_5, x_6, y_6, y_7, x_9]$$

We compute an elimination order GB for the polynomials.

```
> G := Groebner[gbasis]( F, lexdeg( V, [m] ) ):
  map(indets,G);
```

$[\{m\}, \{m, x_9\}, \{m, y_7\}, \{m, y_6\}, \{m, y_6, x_6\}, \{m, y_6, y_5\}, \{m, y_6, x_5\}, \{m, y_4\}, \{m, x_2\}, \{m, y_6\}, \{m, y_6, x_6\}, \{m, y_6, x_5, y_5, x_6\}, \{m, y_6, x_5, y_5\}, \{m, y_6, x_5, y_5, x_6\}, \{m, y_6, x_5, y_5, x_6\}, \{m, y_6, x_5, y_5, x_6\}, \{m, y_6, x_5, y_5, x_6\}]$

The first polynomial in the GB is in m alone (as desired). We factor it.

```
> factor(G[1]);
```

$$m^2 (m - 1)(7 m^4 + 8 m^3 - 20 m^2 + 16 m - 4)$$

Along with the desired result are factors for the solutions $m = 0$, $m = 1$. These are invalid solutions for the Circle Packing problem, so we recompute the GB, this time including the constraint that $m \neq 0$, $m \neq 1$.

```
> G := Groebner[gbasis]( [op(F),1-z*m*(m-1)], lexdeg(
  [op(V),z], [m] ) );
  map(indets,G);
```

$$G := [7 m^4 + 8 m^3 - 20 m^2 + 16 m - 4, 28 z + 77 m^3 + 116 m^2 - 160 m + 156, -1 + x_9 + m, y_7 + m - 1, 16 y_6 + 7 m^3 + 8 m^2 - 18 m, 16 x_6 - 7 m^3 - 8 m^2 + 18 m - 16, 16 y_5 - 7 m^3 - 8 m^2 + 18 m - 16, 16 x_5 + 7 m^3 + 8 m^2 - 18 m, y_4 - m, x_2 - m]$$

$[\{m\}, \{m, z\}, \{m, x_9\}, \{m, y_7\}, \{m, y_6\}, \{m, x_6\}, \{m, y_5\}, \{m, x_5\}, \{m, y_4\}, \{m, x_2\}]$

```
> factor(G[1]);
```

$$7 m^4 + 8 m^3 - 20 m^2 + 16 m - 4$$

This time the GB contains the desired minimal polynomial for m with no extra factors. We can now compute m, the least positive root of this polynomial, and the corresponding radius of the circles.

```
> solve( {G[1], m > 0} , {m} );
  m := subs( allvalues(%), m );
  evalf(m);
```

$$\{m = \mathrm{RootOf}(7\_Z^4 + 8\_Z^3 - 20\_Z^2 + 16\_Z - 4, 0.4195420911)\}$$

$$m := -\frac{2}{7} - \frac{4\sqrt{2}}{7} + \frac{\sqrt{50+44\sqrt{2}}}{7}$$

$$0.419542092$$

```
> r = evalf( m/(2*m+2) );
```

$$r = 0.1477737414$$

✓

This is the correct radius for the first arrangement. Knowing m, we can also use the GB and our original constraints to calculate the exact coordinates of all 10 centre points of the circles. We do this, and plot the arrangement.

```
> coords := solve( {op(G[2..-1]), op(bdry)}, {seq(
  op([x[i],y[i]]), i=1..10 ), z} );
```

$$coords := \{y_{10} = 1, x_7 = 1, x_8 = 0, y_8 = 1, y_9 = 1, x_{10} = 1, x_4 = 0, x_1 = 0, y_1 = 0, y_2 = 0, x_3 = 1,$$

$$y_3 = 0, y_4 = -\frac{2}{7} - \frac{4\sqrt{2}}{7} + \frac{\sqrt{50+44\sqrt{2}}}{7}, x_2 = -\frac{2}{7} - \frac{4\sqrt{2}}{7} + \frac{\sqrt{50+44\sqrt{2}}}{7},$$

$$x_9 = \frac{9}{7} + \frac{4\sqrt{2}}{7} - \frac{\sqrt{50+44\sqrt{2}}}{7}, y_6 = \frac{11}{14} + \frac{\sqrt{2}}{14} - \frac{\sqrt{2}\sqrt{50+44\sqrt{2}}}{28},$$

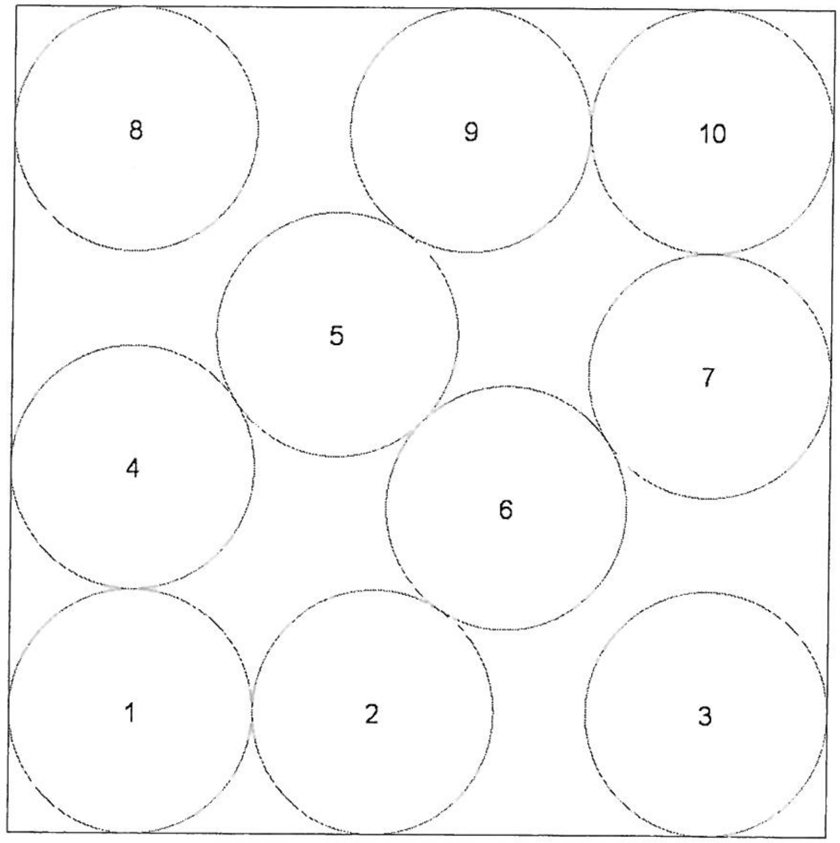
$$z = -2 - \frac{11\sqrt{50+44\sqrt{2}}}{98} - \frac{3\sqrt{2}\sqrt{50+44\sqrt{2}}}{49}, y_5 = \frac{3}{14} - \frac{\sqrt{2}}{14} + \frac{\sqrt{2}\sqrt{50+44\sqrt{2}}}{28},$$

$$x_6 = \frac{3}{14} - \frac{\sqrt{2}}{14} + \frac{\sqrt{2}\sqrt{50+44\sqrt{2}}}{28}, x_5 = \frac{11}{14} + \frac{\sqrt{2}}{14} - \frac{\sqrt{2}\sqrt{50+44\sqrt{2}}}{28},$$

$$y_7 = \frac{9}{7} + \frac{4\sqrt{2}}{7} - \frac{\sqrt{50+44\sqrt{2}}}{7}\}$$

```
> for i to 10 do
    C[i] := plottools[circle]( subs( coords, [x[i],y[i]] ),
  m/2, color=blue ):
  od:
  T := plots[textplot]( subs( coords, [seq( [x[i],y[i],i],
  i=1..10 )] ) ):
  SQ := plottools[rectangle]( [-m/2,-m/2], [1+m/2,1+m/2] ):
  plots[display]( SQ, seq( C[i], i=1..10 ), T,
  scaling=constrained, axes=none, title="Packing A" );
```

Packing A



b)

```
> m := 'm':
```

4

We input the boundary, symmetry and distance equations for the first arrangment. Boundary equations specify which coordinates are touching the outside of the square. In this arrangement, symmetry is found over the diagonal line through [0, 0] and [1, 1] (the line $y = x$) between circle pairs (2, 4), (6, 7), (8, 10) and 5 with itself. Distance equations (with distance m) are specified for every pair of touching circles. When the circles are separated by a line parallel to one of the axes we give a simplified linear constraint rather than the general quadratic constraint.

```
> bdry := [y[1] = 0, y[2] = 0, x[3] = 1, x[4] = 0, x[8] = 1,
  y[9] = 1, y[10] = 1];
  symm := [x[2] = y[4], x[5] = y[5], x[6] = y[7], y[6] = x[7],
  y[8] = x[10]];
  dist := [d2(1,5) = m^2, d2(2,3) = m^2, d2(2,5) = m^2, d2(2,7)
  = m^2, d2(3,7) = m^2, d2(4,5) = m^2, d2(4,6) = m^2, d2(4,9) =
  m^2, d2(5,6) = m^2, d2(5,7) = m^2, d2(6,7) = m^2, d2(6,9) =
  m^2, d2(6,10) = m^2, d2(7,8) = m^2, d2(8,10) = m^2];
```

$$bdry := [y_1 = 0, y_2 = 0, x_3 = 1, x_4 = 0, x_8 = 1, y_9 = 1, y_{10} = 1]$$

$$symm := [x_2 = y_4, x_5 = y_5, x_6 = y_7, y_6 = x_7, y_8 = x_{10}]$$

$$dist := [(x_5-x_1)^2+(y_5-y_1)^2=m^2, (x_3-x_2)^2+(y_3-y_2)^2=m^2,$$
$$(x_5-x_2)^2+(y_5-y_2)^2=m^2, (x_7-x_2)^2+(y_7-y_2)^2=m^2, (x_7-x_3)^2+(y_7-y_3)^2=m^2,$$
$$(x_5-x_4)^2+(y_5-y_4)^2=m^2, (x_6-x_4)^2+(y_6-y_4)^2=m^2, (x_9-x_4)^2+(y_9-y_4)^2=m^2,$$
$$(x_6-x_5)^2+(y_6-y_5)^2=m^2, (x_7-x_5)^2+(y_7-y_5)^2=m^2, (x_7-x_6)^2+(y_7-y_6)^2=m^2,$$
$$(x_9-x_6)^2+(y_9-y_6)^2=m^2, (x_{10}-x_6)^2+(y_{10}-y_6)^2=m^2, (x_8-x_7)^2+(y_8-y_7)^2=m^2,$$
$$(x_{10}-x_8)^2+(y_{10}-y_8)^2=m^2]$$

We construct a polynomial basis from the given equations, along with a list of the appropriate variables.

```
> F := [seq( subs( [op(bdry),op(symm)], lhs(e)-rhs(e) ), e=dist
)];
V := map( z -> `if`( has(F,z), z, NULL ), [seq(
op([x[i],y[i]]), i=1..10 )] );
```

$$F := [(y_5-x_1)^2+y_5^2-m^2, (1-y_4)^2+y_3^2-m^2, (y_5-y_4)^2+y_5^2-m^2,$$
$$(x_7-y_4)^2+y_7^2-m^2, (x_7-1)^2+(y_7-y_3)^2-m^2, (y_5-y_4)^2+y_5^2-m^2,$$
$$(x_7-y_4)^2+y_7^2-m^2, x_9^2+(1-y_4)^2-m^2, (y_7-y_5)^2+(x_7-y_5)^2-m^2,$$
$$(y_7-y_5)^2+(x_7-y_5)^2-m^2, (x_7-y_7)^2+(y_7-x_7)^2-m^2, (x_9-y_7)^2+(1-x_7)^2-m^2,$$
$$(x_{10}-y_7)^2+(1-x_7)^2-m^2, (x_{10}-y_7)^2+(1-x_7)^2-m^2, (x_{10}-1)^2+(1-x_{10})^2-m^2]$$

$$V := [x_1, y_3, y_4, y_5, x_7, y_7, x_9, x_{10}]$$

We compute total degree order GB for the polynomials... (Maple is too slow computing the elimination order GB)

```
> G := Groebner[gbasis]( F, tdeg(op(V),m) ):
map(indets,G);
```

$$[\{x_9, y_3\}, \{m, x_{10}\}, \{m, x_{10}, y_7, x_9\}, \{m, x_{10}, y_4, x_9, x_7, y_5\}, \{m, x_{10}, y_7, x_9, x_7, y_5\},$$
$$\{x_{10}, y_7, x_7\}, \{m, x_{10}, y_7, x_9, x_7, y_5\}, \{m, y_4, y_7, x_9, x_7, y_5\}, \{m, x_{10}, y_7, x_9, x_7, y_5\},$$
$$\{m, x_{10}, y_7, x_9, x_7, y_5\}, \{m, x_{10}, y_4, y_7, x_9, x_7\}, \{m, x_9, y_5\}, \{m, y_4, x_9, y_5\}, \{m, y_4, x_9\},$$
$$\{m, x_1, x_9, y_5\}, \{m, x_{10}, x_9, x_7, y_5\}, \{m, x_{10}, x_9, x_7, y_5\}, \{m, x_{10}, y_7, x_9, x_7, y_5\},$$
$$\{m, x_{10}, x_9, x_7, y_5\}, \{m, x_{10}, x_9, x_7, y_5\}, \{m, x_{10}, y_4, x_9, x_7, y_5\}, \{m, x_{10}, x_9, x_7, y_5\},$$
$$\{m, x_{10}, x_9, x_7, y_5\}, \{x_{10}, x_9, x_7\}, \{m, x_{10}, x_9, x_7, y_5\}, \{m, x_{10}, x_9, x_7, y_5\}]$$

... and use FGLM to compute a lexicographic order GB.

```
> GL := FGLM( G, [op(V),m], LTgrevlex, [op(V),m] )[1]:
map(indets,GL);
```

$$[\{m\}, \{m, x_{10}\}, \{m, x_{10}\}, \{m, x_9\}, \{m, x_{10}, x_9\}, \{m, x_9\}, \{m, y_7\}, \{m, x_{10}, y_7, x_9\},$$
$$\{m, x_{10}, y_7, x_9\}, \{m, x_{10}, y_7, x_9\}, \{m, x_7\}, \{m, x_{10}, x_9, x_7\}, \{x_{10}, x_9, x_7\}, \{x_{10}, y_7, x_7\},$$

$\{m, x_{10}, y_7, x_7\}, \{m, y_5\}, \{m, x_{10}, y_7, x_9, x_7, y_5\}, \{m, x_{10}, y_7, x_9, x_7, y_5\},$

$\{m, x_{10}, y_7, x_9, x_7, y_5\}, \{m, x_{10}, y_7, x_9, x_7, y_5\}, \{m, x_{10}, y_7, x_9, x_7, y_5\}, \{m, y_4\},$

$\{m, x_{10}, y_4, y_7, x_9, x_7, y_5\}, \{m, x_{10}, y_4, y_7, x_9, x_7, y_5\}, \{m, x_{10}, y_4, y_7, x_9, x_7\},$

$\{m, x_{10}, y_4, y_7, x_9, x_7, y_5\}, \{m, y_4, x_9\}, \{x_9, y_3\}, \{m, x_1, x_{10}, y_7, x_9, x_7, y_5\}]$

The first polynomial in the GB is in m alone (as desired). We factor it.

```
> factor(GL[1]);
```

$$m^2 (121 m^4 - 112 m^2 + 16)$$

Along with the desired result are factors for the solution $m = 0$. This is an invalid solution for the Circle Packing problem, so we recompute the GB, this time including the constraint that $m \neq 0$.

```
> G := Groebner[gbasis]( [op(F),1-z*m], tdeg(op(V),z,m) ):
  GL := FGLM( G, [op(V),z,m], LTgrevlex, [op(V),z,m] )[1];
  map(indets,GL);
```

$GL := [121 m^4 - 112 m^2 + 16, 16 z + 121 m^3 - 112 m, 11 m^2 + 20 x_{10} - 16,$

$20 x_9 - 8 + 33 m^2, 10 y_7 - 6 + 11 m^2, 20 x_7 - 16 + 11 m^2, 20 y_5 - 4 - 11 m^2,$

$10 y_4 - 4 - 11 m^2, 20 y_3 - 8 + 33 m^2, 10 x_1^{\,2} - 11 x_1 m^2 - 4 x_1]$

$[\{m\}, \{m, z\}, \{m, x_{10}\}, \{m, x_9\}, \{m, y_7\}, \{m, x_7\}, \{m, y_5\}, \{m, y_4\}, \{m, y_3\}, \{m, x_1\}$

$]$

```
> factor(GL[1]);
```

$$121 m^4 - 112 m^2 + 16$$

This time the GB contains the desired minimal polynomial for m with no extra factors. We can now compute m, the least positive root of this polynomial, and the corresponding radius of the circles.

```
> solve( {GL[1], m > 0} , {m} );
  m := min( seq( subs(allvalues(e),m), e=% ) );
  evalf(m);
```

$\{m = \mathrm{RootOf}(-\mathrm{RootOf}(121\,\_Z^2 + 16 - 112\,\_Z, 0.1765205277) + \_Z^2, 0.4201434609)\},$

$\{m = \mathrm{RootOf}(-\mathrm{RootOf}(121\,\_Z^2 + 16 - 112\,\_Z, 0.7490993070) + \_Z^2, 0.8655052322)\}$

$$m := \frac{5\sqrt{2}}{11} - \frac{\sqrt{6}}{11}$$

$$0.4201434606$$

```
> r = evalf( m/(2*m+2) );
```

$$r = 0.1479228938$$

This is the correct radius for the second arrangement. Knowing m, we can also use the GB and our original constraints to calculate the exact coordinates of all 10 centre points of the circles.

```
> coords := solve( {op(G[2..-1]), op(bdry), op(symm),
```

```
op(dist)}, {seq( op([x[i],y[i]]), i=1..10 ), z} );
```

$coords := \{x_5 = \frac{5}{11} - \frac{\sqrt{2}\sqrt{6}}{22}, z = \frac{5\sqrt{2}}{4} + \frac{\sqrt{6}}{4}, x_{10} = \frac{6}{11} + \frac{\sqrt{2}\sqrt{6}}{22}, x_7 = \frac{6}{11} + \frac{\sqrt{2}\sqrt{6}}{22},$

$y_3 = -\frac{4}{11} + \frac{3\sqrt{2}\sqrt{6}}{22}, y_{10} = 1, y_9 = 1, x_4 = 0, x_1 = 0, y_1 = 0, y_2 = 0, x_3 = 1,$

$y_4 = \frac{10}{11} - \frac{\sqrt{2}\sqrt{6}}{11}, x_8 = 1, x_9 = -\frac{4}{11} + \frac{3\sqrt{2}\sqrt{6}}{22}, y_7 = \frac{1}{11} + \frac{\sqrt{2}\sqrt{6}}{11}, y_6 = \frac{6}{11} + \frac{\sqrt{2}\sqrt{6}}{22},$

$y_8 = \frac{6}{11} + \frac{\sqrt{2}\sqrt{6}}{22}, x_2 = \frac{10}{11} - \frac{\sqrt{2}\sqrt{6}}{11}, y_5 = \frac{5}{11} - \frac{\sqrt{2}\sqrt{6}}{22}, x_6 = \frac{1}{11} + \frac{\sqrt{2}\sqrt{6}}{11}\}, \{$

$x_5 = \frac{5}{11} - \frac{\sqrt{2}\sqrt{6}}{22}, x_1 = \frac{10}{11} - \frac{\sqrt{2}\sqrt{6}}{11}, z = \frac{5\sqrt{2}}{4} + \frac{\sqrt{6}}{4}, x_{10} = \frac{6}{11} + \frac{\sqrt{2}\sqrt{6}}{22},$

$x_7 = \frac{6}{11} + \frac{\sqrt{2}\sqrt{6}}{22}, y_3 = -\frac{4}{11} + \frac{3\sqrt{2}\sqrt{6}}{22}, y_{10} = 1, y_9 = 1, x_4 = 0, y_1 = 0, y_2 = 0, x_3 = 1,$

$y_4 = \frac{10}{11} - \frac{\sqrt{2}\sqrt{6}}{11}, x_8 = 1, x_9 = -\frac{4}{11} + \frac{3\sqrt{2}\sqrt{6}}{22}, y_7 = \frac{1}{11} + \frac{\sqrt{2}\sqrt{6}}{11}, y_6 = \frac{6}{11} + \frac{\sqrt{2}\sqrt{6}}{22},$

$y_8 = \frac{6}{11} + \frac{\sqrt{2}\sqrt{6}}{22}, x_2 = \frac{10}{11} - \frac{\sqrt{2}\sqrt{6}}{11}, y_5 = \frac{5}{11} - \frac{\sqrt{2}\sqrt{6}}{22}, x_6 = \frac{1}{11} + \frac{\sqrt{2}\sqrt{6}}{11}\}$

We get two solutions! They agree completely except in the x-coordinate of the first circle:

```
> coords[1] minus coords[2], coords[2] minus coords[1];
```

$$\{x_1 = 0\}, \{x_1 = \frac{10}{11} - \frac{\sqrt{2}\sqrt{6}}{11}\}$$

Interesting

Does this mean that there are two possible placements for the first circle that both give a packing with the desired circle radius? Closer inspection shows that $x_1 = x_2$ in the second solution, thus placing circle 1 on top of circle 2. This is no good. We add the additional constraint that $x_1 \neq y_4$ (since $x_2 = y_4$ from symmetry) and recompute from the GB onwards.

```
> m := 'm':
  G := Groebner[gbasis]( [op(F),1-z*m*(x[1]-y[4])],
  tdeg(op(V),z,m) ):
  GL := FGLM( G, [op(V),z,m], LTgrevlex, [op(V),z,m] )[1];
  map(indets,GL);
  factor(GL[1]);
```

$GL := [121\, m^4 - 112\, m^2 + 16, 320\, z - 4719\, m^3 + 3884\, m, 11\, m^2 + 20\, x_{10} - 16,$

$20\, x_9 - 8 + 33\, m^2, 10\, y_7 - 6 + 11\, m^2, 20\, x_7 - 16 + 11\, m^2, 20\, y_5 - 4 - 11\, m^2,$

$10\, y_4 - 4 - 11\, m^2, 20\, y_3 - 8 + 33\, m^2, x_1]$

$$[\{m\}, \{m, z\}, \{m, x_{10}\}, \{m, x_9\}, \{m, y_7\}, \{m, x_7\}, \{m, y_5\}, \{m, y_4\}, \{m, y_3\}, \{x_1\}]$$

$$121\, m^4 - 112\, m^2 + 16$$

```
> solve( {GL[1], m > 0} , {m} );
  m := min( seq( subs(allvalues(e),m), e=% ) );
  evalf(m);
  r = evalf( m/(2*m+2) );
```

$$\{m = \mathrm{RootOf}(-\mathrm{RootOf}(121\ \_Z^2 + 16 - 112\ \_Z, 0.1765205277) + \_Z^2, 0.4201434609)\},$$
$$\{m = \mathrm{RootOf}(-\mathrm{RootOf}(121\ \_Z^2 + 16 - 112\ \_Z, 0.7490993070) + \_Z^2, 0.8655052322)\}$$

$$m := \frac{5\sqrt{2}}{11} - \frac{\sqrt{6}}{11}$$

$$0.4201434606$$

$$r = 0.1479228938$$

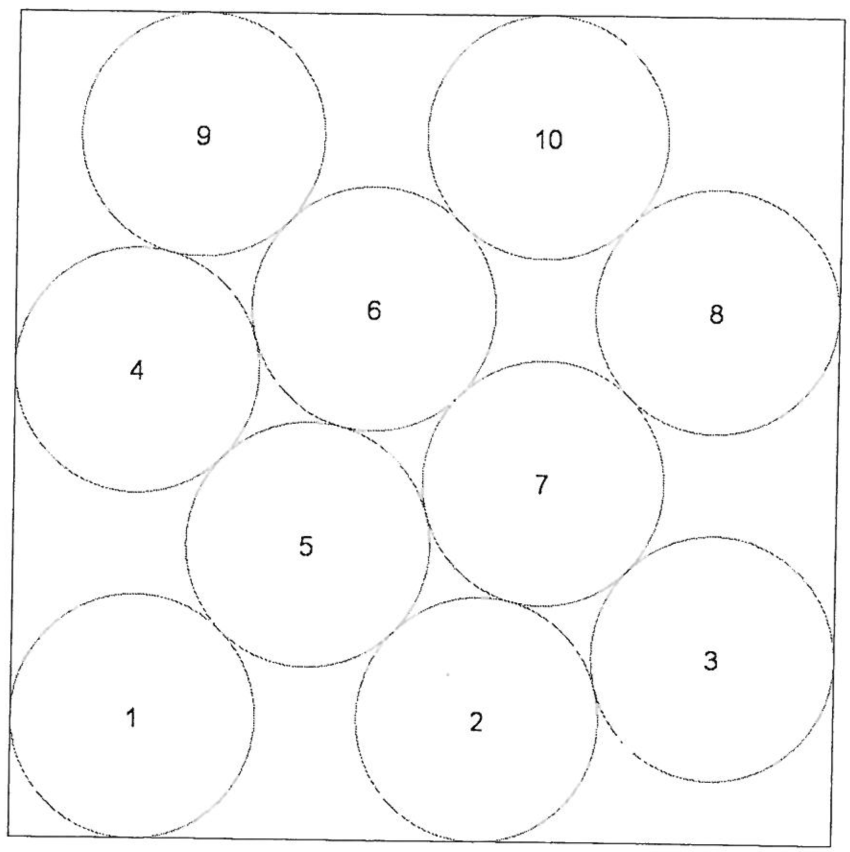
We still get the correct radius for this arrangment. We recompute the coordinates of all 10 centre points, this time getting a unique solution, and plot the arrangement.

```
> coords := solve( {op(G[2..-1]), op(bdry), op(symm),
  op(dist)}, {seq( op([x[i],y[i]]), i=1..10 ), z} );
```

$$coords := \{x_5 = \frac{5}{11} - \frac{\sqrt{2}\sqrt{6}}{22}, x_{10} = \frac{6}{11} + \frac{\sqrt{2}\sqrt{6}}{22}, x_7 = \frac{6}{11} + \frac{\sqrt{2}\sqrt{6}}{22},$$
$$y_3 = -\frac{4}{11} + \frac{3\sqrt{2}\sqrt{6}}{22}, y_{10} = 1, y_9 = 1, x_4 = 0, x_1 = 0, y_1 = 0, y_2 = 0, x_3 = 1,$$
$$y_4 = \frac{10}{11} - \frac{\sqrt{2}\sqrt{6}}{11}, x_8 = 1, x_9 = -\frac{4}{11} + \frac{3\sqrt{2}\sqrt{6}}{22}, y_7 = \frac{1}{11} + \frac{\sqrt{2}\sqrt{6}}{11}, y_6 = \frac{6}{11} + \frac{\sqrt{2}\sqrt{6}}{22},$$
$$z = -\frac{7\sqrt{2}}{4} - \frac{5\sqrt{6}}{8}, y_8 = \frac{6}{11} + \frac{\sqrt{2}\sqrt{6}}{22}, x_2 = \frac{10}{11} - \frac{\sqrt{2}\sqrt{6}}{11}, y_5 = \frac{5}{11} - \frac{\sqrt{2}\sqrt{6}}{22},$$
$$x_6 = \frac{1}{11} + \frac{\sqrt{2}\sqrt{6}}{11}\}$$

```
> for i to 10 do
    C[i] := plottools[circle]( subs( coords, [x[i],y[i]] ),
  m/2, color=blue ):
  od:
  T := plots[textplot]( subs( coords, [seq( [x[i],y[i],i],
  i=1..10 )] ) ):
  SQ := plottools[rectangle]( [-m/2,-m/2], [1+m/2,1+m/2] ):
  plots[display]( SQ, seq( C[i], i=1..10 ), T,
  scaling=constrained, axes=none, title="Packing B" );
```

Packing B



Note that we have shown that $x_1 = 0$ must be true, which was not specified in the original diagram, but follows from the claim that this arrangement is symmetric.

c)

```
> m := 'm':
```

4

We input the boundary, symmetry and distance equations for the first arrangment. Boundary equations specify which coordinates are touching the outside of the square. In this arrangement, symmetry is found with circle 6 over the vertical line midway between circles 2 and 3, and over the horizontal line midway between circles 3 and 7. Distance equations (with distance m) are specified for every pair of touching circles. When the circles are separated by a line parallel to one of the axes we give a simplified linear constraint rather than the general quadratic constraint.

Note: The only way that we were able to get a unique (in fact finite) set of solutions for this arrangement was to include the constraint that circles 5 and 6 must be touching, which was not specified in the original diagram..

Free circles: In the arrangement, circle 8 is free. We fix it to lie touching circle 9 and the left edge of the square so as to provide a unique solution for the position of every circle.

```
> bdry := [x[1] = 0, y[1] = 0, y[2] = 0, x[3] = 1, y[3] = 0,
  x[4] = 0, x[7] = 1, x[8] = 0, y[9] = 1, y[10] = 1];
  symm := [y[4] = x[2], x[6] = (x[2]+x[3])/2, y[6] =
  (y[3]+y[7])/2];
```

```
dist := [x[2] = x[1]+m, y[4] = y[1]+m, d2(2,6) = m^2, d2(3,6)
= m^2, d2(4,5) = m^2, d2(5,6) = m^2, d2(5,9) = m^2, d2(6,7) =
m^2, d2(7,10) = m^2, d2(8,9) = m^2, x[10] = x[9]+m];
```

$$bdry := [x_1 = 0, y_1 = 0, y_2 = 0, x_3 = 1, y_3 = 0, x_4 = 0, x_7 = 1, x_8 = 0, y_9 = 1, y_{10} = 1]$$

$$symm := \left[y_4 = x_2, x_6 = \frac{1}{2}x_2 + \frac{1}{2}x_3, y_6 = \frac{1}{2}y_3 + \frac{1}{2}y_7 \right]$$

$$dist := [x_2 = x_1 + m, y_4 = y_1 + m, (x_6 - x_2)^2 + (y_6 - y_2)^2 = m^2, (x_6 - x_3)^2 + (y_6 - y_3)^2 = m^2, (x_5 - x_4)^2 + (y_5 - y_4)^2 = m^2, (x_6 - x_5)^2 + (y_6 - y_5)^2 = m^2, (x_9 - x_5)^2 + (y_9 - y_5)^2 = m^2, (x_7 - x_6)^2 + (y_7 - y_6)^2 = m^2, (x_{10} - x_7)^2 + (y_{10} - y_7)^2 = m^2, (x_9 - x_8)^2 + (y_9 - y_8)^2 = m^2, x_{10} = x_9 + m]$$

We construct a polynomial basis from the given equations, along with a list of the appropriate variables.

```
> F := [seq( subs( bdry, lhs(e)-rhs(e) ), e=[op(symm),op(dist)]
)];
V := map( z -> `if`( has(F,z), z, NULL ), [seq(
op([x[i],y[i]]), i=1..10 )] );
```

$$F := \left[y_4 - x_2, x_6 - \frac{1}{2}x_2 - \frac{1}{2}, y_6 - \frac{1}{2}y_7, x_2 - m, y_4 - m, (x_6 - x_2)^2 + y_6^2 - m^2, (x_6 - 1)^2 + y_6^2 - m^2, x_5^2 + (y_5 - y_4)^2 - m^2, (x_6 - x_5)^2 + (y_6 - y_5)^2 - m^2, (x_9 - x_5)^2 + (1 - y_5)^2 - m^2, (1 - x_6)^2 + (y_7 - y_6)^2 - m^2, (x_{10} - 1)^2 + (1 - y_7)^2 - m^2, x_9^2 + (1 - y_8)^2 - m^2, x_{10} - x_9 - m \right]$$

$$V := [x_2, y_4, x_5, y_5, x_6, y_6, y_7, y_8, x_9, x_{10}]$$

We compute an elimination order GB for the polynomials.

```
> G := Groebner[gbasis]( F, lexdeg( V, [m] ) ):
map(indets,G);
```

$$[\{m\}, \{m, x_{10}\}, \{m, x_{10}, x_9\}, \{m, y_7\}, \{y_7, y_6\}, \{m, x_6\}, \{m, y_5\}, \{m, x_5\}, \{m, y_4\}, \{m, x_2\}, \{m, x_{10}, y_7\}, \{m, x_{10}, y_7, x_5, y_5\}, \{m, x_{10}, y_7, x_5, y_5\}, \{m, x_{10}, y_7, y_8\}, \{m, y_7\}, \{m, y_7, x_5, y_5\}, \{m, y_7, x_5, y_5\}, \{m, y_5\}, \{m, x_5, y_5\}]$$

The first polynomial in the GB is in m alone (as desired). We factor it.

```
> factor(G[1]);
```

$$m\,(m-1)\,(1062\,m^{14} + 7476\,m^{13} + 20848\,m^{12} + 24256\,m^{11} - 8144\,m^{10} - 52494\,m^9 - 38026\,m^8 + 29798\,m^7 + 46489\,m^6 - 7202\,m^5 - 25099\,m^4 + 3856\,m^3 + 6720\,m^2 - 2816\,m + 320)$$

Along with the desired result are factors for the solutions $m = 0$, $m = 1$. These are invalid solutions for the Circle Packing problem, so we recompute a GB (this time w.r.t. lexicographic order with FGLM), this time including the constraint that $m \neq 0$, $m \neq 1$.

```
> G := Groebner[gbasis]( [op(F),1-z*m*(m-1)], tdeg(op(V),z,m)
  ):
  GL := FGLM( G, [op(V),z,m], LTgrevlex, [op(V),z,m] )[1]:
  map(indets,GL);
```

$$[\{m\}, \{m, z\}, \{m, x_{10}\}, \{m, x_9\}, \{m, y_8\}, \{m, y_7\}, \{m, y_6\}, \{m, x_6\}, \{m, y_5\}, \{m, x_5\}, \{m, y_4\}, \{m, x_2\}]$$

```
> factor(GL[1]);
```

$$1062\, m^{14} + 7476\, m^{13} + 20848\, m^{12} + 24256\, m^{11} - 8144\, m^{10} - 52494\, m^9 - 38026\, m^8 + 29798\, m^7 + 46489\, m^6 - 7202\, m^5 - 25099\, m^4 + 3856\, m^3 + 6720\, m^2 - 2816\, m + 320$$

This time the GB contains the desired minimal polynomial for m with no extra factors. We can now compute m, the least positive root of this polynomial, and the corresponding radius of the circles.

```
> solve( {GL[1], m > 0}, {m} ):
  m := min( seq( subs(allvalues(e),m), e=% ) );
  evalf(m);
```

$$m := \mathrm{RootOf}(1062\,\_Z^{14} + 7476\,\_Z^{13} + 20848\,\_Z^{12} + 24256\,\_Z^{11} - 8144\,\_Z^{10} - 52494\,\_Z^9 - 38026\,\_Z^8 + 29798\,\_Z^7 + 46489\,\_Z^6 - 7202\,\_Z^5 - 25099\,\_Z^4 + 3856\,\_Z^3 + 6720\,\_Z^2 - 2816\,\_Z + 320, 0.4211897032)$$

$$0.4211897032$$

```
> r = evalf( m/(2*m+2) );
```

$$r = 0.1481820838$$

This is the correct radius for the third arrangement. Knowing m, we can also use the GB and our original constraints to calculate the exact coordinates of all 10 centre points of the circles. We do this, and plot the arrangement.

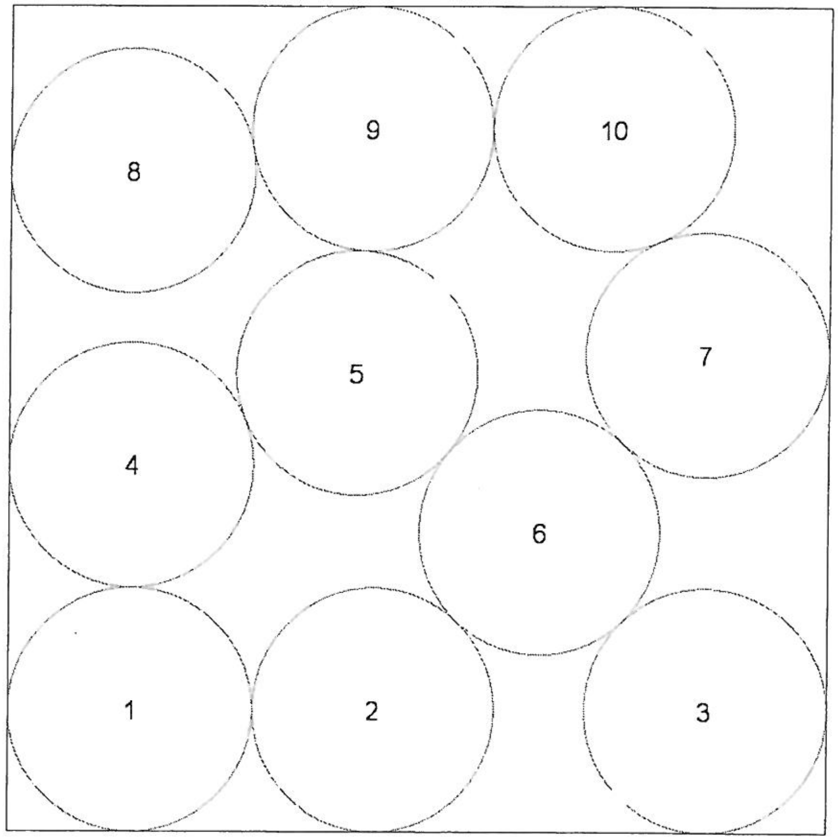
```
> coords := solve( {op(GL[2..-1]), op(bdry), op(symm),
  op(dist)}, {seq( op([x[i],y[i]]), i=1..10 ), z} ):
  evalf(coords);
```

$$\{x_5 = 0.39029481, y_7 = 0.61203078, x_8 = 0., x_7 = 1., y_3 = 0., y_4 = 0.4211897032, x_6 = 0.7105948516, y_{10} = 1., y_9 = 1., x_4 = 0., x_1 = 0., y_1 = 0., y_2 = 0., x_3 = 1., x_9 = 0.4148570, x_2 = 0.4211897032, x_{10} = 0.8360467, y_8 = 0.9272374277, y_6 = 0.30601541, y_5 = 0.57952711, z = -4.101908851\}$$

```
> for i to 10 do
    C[i] := plottools[circle]( subs( coords, [x[i],y[i]] ),
  m/2, color=blue ):
  od:
  T := plots[textplot]( subs( coords, [seq( [x[i],y[i],i],
  i=1..10 )] ) ):
```

```
SQ := plottools[rectangle]( [-m/2,-m/2], [1+m/2,1+m/2] ):
plots[display]( SQ, seq( C[i], i=1..10 ), T,
scaling=constrained, axes=none, title="Packing C" );
```

Packing C



d)

```
> m := 'm':
```

4

We input the boundary, symmetry and distance equations for the first arrangment. Boundary equations specify which coordinates are touching the outside of the square. In this arrangement, symmetry is found with circle 6 over the vertical line midway between circles 8 and 9, and over the horizontal line midway between circles 4 and 8. Symmetry is also found in the tightly packed parallelogram formation of circles 2, 3, 6 & 7, whose sides must be parallel. Distance equations (with distance *m*) are specified for every pair of touching circles. When the circles are separated by a line parallel to one of the axes we give a simplified linear constraint rather than the general quadratic constraint.

```
> bdry := [x[1] = 0, y[2] = 0, y[3] = 0, x[4] = 0, x[7] = 1,
  x[8] = 0, y[8] = 1, y[9] = 1, x[10] = 1];
  symm := [x[5] = (x[8]+x[9])/2, y[5] = (y[4]+y[8])/2,
  x[6]-x[2] = x[7]-x[3], y[6]-y[2] = y[7]-y[3]];
  dist := [d2(1,2) = m^2, y[4] = y[1]+m, x[3] = x[2]+m, d2(2,6)
  = m^2, d2(3,7) = m^2, d2(4,5) = m^2, d2(5,6) = m^2, d2(5,8) =
  m^2, d2(5,9) = m^2, d2(6,7) = m^2, y[10] = y[7]+m, d2(9,10) =
```

```
m^2];
```

$$bdry := [x_1 = 0, y_2 = 0, y_3 = 0, x_4 = 0, x_7 = 1, x_8 = 0, y_8 = 1, y_9 = 1, x_{10} = 1]$$

$$symm := \left[x_5 = \frac{1}{2}x_8 + \frac{1}{2}x_9, y_5 = \frac{1}{2}y_4 + \frac{1}{2}y_8, x_6 - x_2 = x_7 - x_3, y_6 - y_2 = y_7 - y_3\right]$$

$$dist := [(x_2 - x_1)^2 + (y_2 - y_1)^2 = m^2, y_4 = y_1 + m, x_3 = x_2 + m, (x_6 - x_2)^2 + (y_6 - y_2)^2 = m^2,$$
$$(x_7 - x_3)^2 + (y_7 - y_3)^2 = m^2, (x_5 - x_4)^2 + (y_5 - y_4)^2 = m^2, (x_6 - x_5)^2 + (y_6 - y_5)^2 = m^2,$$
$$(x_8 - x_5)^2 + (y_8 - y_5)^2 = m^2, (x_9 - x_5)^2 + (y_9 - y_5)^2 = m^2, (x_7 - x_6)^2 + (y_7 - y_6)^2 = m^2,$$
$$y_{10} = y_7 + m, (x_{10} - x_9)^2 + (y_{10} - y_9)^2 = m^2]$$

We construct a polynomial basis from the given equations, along with a list of the appropriate variables.

```
> F := [seq( subs( bdry, lhs(e)-rhs(e) ), e=[op(symm),op(dist)]
  )];
  V := map( z -> `if`( has(F,z), z, NULL ), [seq(
  op([x[i],y[i]]), i=1..10 )] );
```

$$F := \left[x_5 - \frac{1}{2}x_9, y_5 - \frac{1}{2}y_4 - \frac{1}{2}, x_6 - x_2 - 1 + x_3, y_6 - y_7, x_2^2 + y_1^2 - m^2, y_4 - y_1 - m,\right.$$
$$x_3 - x_2 - m, (x_6 - x_2)^2 + y_6^2 - m^2, (1 - x_3)^2 + y_7^2 - m^2, x_5^2 + (y_5 - y_4)^2 - m^2,$$
$$(x_6 - x_5)^2 + (y_6 - y_5)^2 - m^2, x_5^2 + (1 - y_5)^2 - m^2, (x_9 - x_5)^2 + (1 - y_5)^2 - m^2,$$
$$\left.(1 - x_6)^2 + (y_7 - y_6)^2 - m^2, y_{10} - y_7 - m, (1 - x_9)^2 + (y_{10} - 1)^2 - m^2\right]$$

$$V := [y_1, x_2, x_3, y_4, x_5, y_5, x_6, y_6, y_7, x_9, y_{10}]$$

We compute an elimination order GB for the polynomials.

```
> G := Groebner[gbasis]( F, lexdeg( V, [m] ) ):
  map(indets,G);
```

$$[\{m\}, \{m, y_{10}\}, \{m, x_9\}, \{m, y_{10}, y_7\}, \{m, y_{10}, y_6\}, \{m, x_6\}, \{m, y_5\}, \{x_9, x_5\}, \{y_4, y_5\},$$
$$\{m, y_{10}, x_9, x_3, y_5\}, \{m, x_2, y_{10}, x_9, y_5\}, \{m, y_1, y_5\}, \{m, y_{10}, x_9, y_5\}, \{m, y_{10}, x_9\},$$
$$\{m, y_{10}, x_9, y_5\}, \{m, y_{10}, x_9, y_5\}]$$

The first polynomial in the GB is in *m* alone (as desired). We factor it.

```
> factor(G[1]);
```

$$m\,(1180129\,m^{18} - 11436428\,m^{17} + 98015844\,m^{16} - 462103584\,m^{15} + 1145811528\,m^{14}$$
$$- 1398966480\,m^{13} + 227573920\,m^{12} + 1526909568\,m^{11} - 1038261808\,m^{10}$$
$$- 2960321792\,m^9 + 7803109440\,m^8 - 9722063488\,m^7 + 7918461504\,m^6$$
$$- 4564076288\,m^5 + 1899131648\,m^4 - 563649536\,m^3 + 114038784\,m^2 - 14172160\,m$$
$$+ 819200)$$

Along with the desired result is a factor for the solution $m = 0$. This is an invalid solution for the Circle Packing problem, so we recompute the GB, this time including the constraint that $m \neq 0$.

```
> G := Groebner[gbasis]( [op(F),1-z*m], lexdeg( [op(V),z], [m]
  ) ):
  map(indets,G);
```

$$[\{m\}, \{m, z\}, \{m, y_{10}\}, \{m, x_9\}, \{m, y_7\}, \{m, y_6\}, \{m, x_6\}, \{m, y_5\}, \{m, x_5\}, \{m, y_4\}, \{m, x_3\}, \{m, x_2\}, \{m, y_1\}]$$

```
> factor(G[1]);
```

$$1180129\, m^{18} - 11436428\, m^{17} + 98015844\, m^{16} - 462103584\, m^{15} + 1145811528\, m^{14} - 1398966480\, m^{13} + 227573920\, m^{12} + 1526909568\, m^{11} - 1038261808\, m^{10} - 2960321792\, m^9 + 7803109440\, m^8 - 9722063488\, m^7 + 7918461504\, m^6 - 4564076288\, m^5 + 1899131648\, m^4 - 563649536\, m^3 + 114038784\, m^2 - 14172160\, m + 819200$$

This time the GB contains the desired minimal polynomial for m with no extra factors. We can now compute m, the least positive root of this polynomial, and the corresponding radius of the circles. We crank up the precision so as to avoid roundoff error when working numerically with polynomials of such high degree.

```
> Digits := 30:
  solve( {G[1], m > 0} , {m} ):
  m := min( seq( subs(allvalues(e),m), e=% ) );
  evalf(m);
```

$$m := \mathrm{RootOf}(1180129\, \_Z^{18} - 11436428\, \_Z^{17} + 98015844\, \_Z^{16} - 462103584\, \_Z^{15} + 1145811528\, \_Z^{14} - 1398966480\, \_Z^{13} + 227573920\, \_Z^{12} + 1526909568\, \_Z^{11} - 1038261808\, \_Z^{10} - 2960321792\, \_Z^9 + 7803109440\, \_Z^8 - 9722063488\, \_Z^7 + 7918461504\, \_Z^6 - 4564076288\, \_Z^5 + 1899131648\, \_Z^4 - 563649536\, \_Z^3 + 114038784\, \_Z^2 - 14172160\, \_Z + 819200, 0.421279543983903432768821760650)$$

$$0.421279543983903432768821760650$$

```
> r = evalf( m/(2*m+2) );
```

$$r = 0.148204322565228798668007362743$$

This is the correct radius for the fourth arrangement. Knowing m, we can also use the GB and our original constraints to calculate the exact coordinates of all 10 centre points of the circles. We do this, and plot the arrangement.

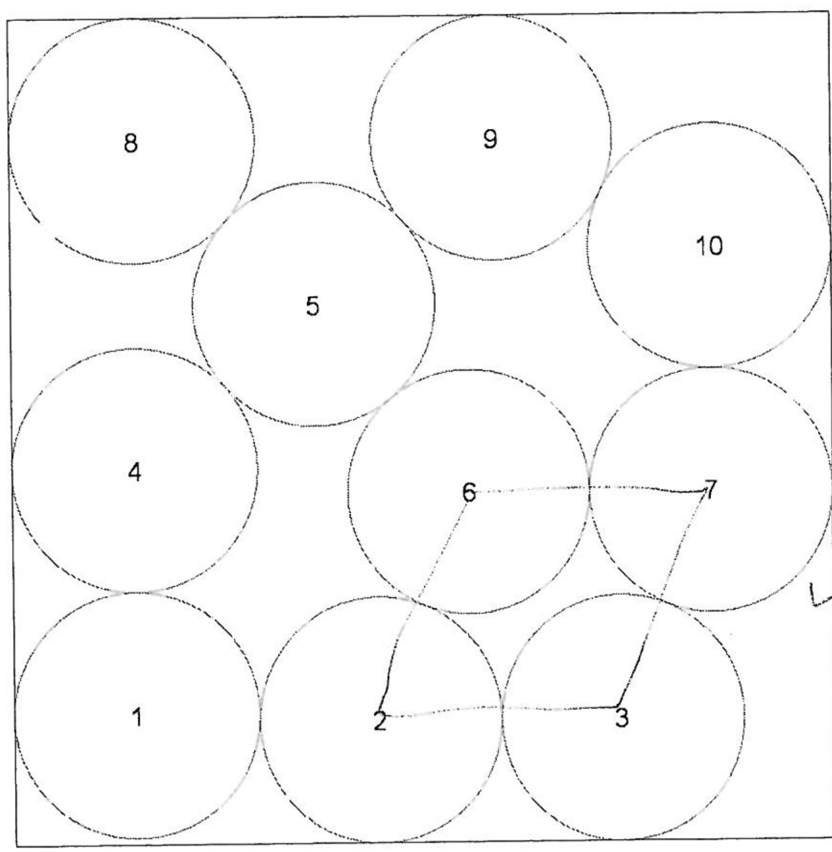
```
> coords := solve( {op(G[2..-1]), op(bdry), op(symm),
  op(dist)}, {seq( op([x[i],y[i]]), i=1..10 ), z} ):
  evalf(coords);
```

$$\{y_2 = 0., y_5 = 0.716378645305845674367048, x_1 = 0., y_9 = 1., x_4 = 0., y_1 = 0.0114777466277879159647O, z = 2.3737207616190563381056184260, y_3 = 0.,$$

$x_5 = 0.31150502618856462212342 9, x_8 = 0., x_7 = 1., y_{10} = 0.81197065768718226140079,$
$y_4 = 0.43275729061169134873350, y_8 = 1., x_{10} = 1., y_6 = 0.39069111370327882863199,$
$x_2 = 0.42112315955268231972 31, y_7 = 0.39069111370327882863199,$
$x_9 = 0.62301005237712924424616, x_3 = 0.84240270353658575249 11,$
$x_6 = 0.57872045601609656723117823 9350\}$

```
> for i to 10 do
    C[i] := plottools[circle]( subs( coords, [x[i],y[i]] ),
  m/2, color=blue ):
  od:
  T := plots[textplot]( subs( coords, [seq( [x[i],y[i],i],
  i=1..10 )] ) ):
  SQ := plottools[rectangle]( [-m/2,-m/2], [1+m/2,1+m/2] ):
  plots[display]( SQ, seq( C[i], i=1..10 ), T,
  scaling=constrained, axes=none, title="Packing D" );
```

Packing D



Scott Cowan
200034814
scottc@sfu.ca

\>