# MATH 800 Assignment 1, Fall 2023

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This assignment is worth 15% of the course grade.
Please hand it in by 11pm Monday September 25th.
Late Penalty $-20\%$ off for up to 36 hours late. Zero after that.
For Maple problems, please export your Maple worksheet to a .pdf file.
If you are registered for MATH 800 please upload your work to Crowdmark.
If you are not registered you will need to Email me your work.

## Question 1 : Using Maple as a calculator. [18 marks]

(a) Using Maple's integration command `int` calculate the following antiderivatives

$$\int x(1-x)dx \quad \int x^2 e^{-x} dx \quad \text{and} \quad \int 4\sqrt{1-x^2}dx.$$

Now calculate the following definite integrals.

$$\int_0^1 x(1-x)dx \quad \int_0^\infty x^2 e^{-x} dx \quad \text{and} \quad \int_0^1 4\sqrt{1-x^2}dx.$$

(b) Using Maple's differentiation command `diff` and evaluation command `eval` calculate $f(0)$, $f'(x)$, $f'(0)$, $f''(x)$ and $f''(0)$ where $f(x) = \sin(x) + \cos(x)$.

(c) Using the Maple `sum` and `factor` commands, in a Maple for loop, calculate and factor the sums

$$\sum_{i=1}^{n} i^k \quad \text{for} \quad 1 \le k \le 6.$$

Note $n$ is a parameter here so you will get formulas (polynomials) in terms of $n$. E.g. for $k = 1$ we have $\sum_{i=1}^{n} i = 1 + 2 + 3 + \ldots + n = \frac{1}{2}n(n+1)$.

(d) Using a Maple while loop and the `isprime` command, find the first prime $p$ greater than 1000.

(e) Using a Maple while loop and the `nextprime` and `irem` commands, find the first prime $p$ greater than 1000 such that $p - 1$ is divisible by 64.

## Question 2 : Programming in Maple. [15 marks]

I want you to learn to program with arrays in Maple. A one dimensional array with $n$ elements indexed from 1 is created with the `Array(1..n)` command. The entries are automatically initialized to 0.

Suppose we have a polynomial $f$ in $x$ of degree $d$ with integer coefficients. One way to represent the polynomial on a computer is with an array $A$ of coefficients indexed from 0 to $d$ where $A[i]$ is the coefficient of $f$ in $x^i$. For example, given $f = 3x^3 - 5x + 6$ we may represent it as the array $A = [6, -5, 0, 3]$. Below I have written two Maple procedures which convert a Maple polynomial $f$ into it's array of coefficients and back.

```
> Maple2Array := proc(f::polynom,x::name)
  local d,A,i;
     if f=0 then return Array(0..-1); fi; # the empty array
     d := degree(f,x);
     A := Array(0..d);
     for i from 0 to d do A[i] := coeff(f,x,i); od;
     A;
  end:
  Array2Maple := proc(A::Array,x::name)
  local r,d,i;
     r := [op(2,A)]; # range(s) for subscripts
     if nops(r)<>1 or lhs(r[1])<>0 then
        error "Array must be one dimensional and indexed from 0"
     fi;
     d := rhs(r[1]); # upper index
     add( A[i]*x^i, i=0..d );
  end:
```

Here is how these work

```
> f := 3*x^3-5*x+6;
```

$$f := 3x^3 - 5x + 6$$

```
> A := Maple2Array(f,x);
```

$$A := Array(0..3, [6, -5, 0, 3])$$

```
> g := Array2Maple(A,y);
```

$$g := 3y^3 - 5y + 6$$

In the following $A$ is an Array representing a polynomial $f$.

(a) Write a Maple procedure DEGREE such that DEGREE($A$) returns the degree of the polynomial stored in $A$.

(b) Write a Maple procedure COEFF such that COEFF($A$,$i$) returns the coefficient of the term of degree $x^i$ of the polynomial stored in $A$.

(c) Write a Maple procedure DIFF such that DIFF($A$) returns an Array containing the derivative of the polynomial stored in $A$.

(d) Write a Maple procedure EVAL such that EVAL($A, z$) returns the value $f(z)$.

For $f = 3x^3 - 5x + 6$ and $A = Maple2Array(f, x)$ test your procedures on DEGREE($A$), COEFF($A$,3), COEFF($A$,6), EVAL($A$,2), DIFF($A$), DEGREE(DIFF($A$)) and other test examples of your choice.

## Question 3 : Analysis of Algorithms [15 marks]

(a) For a constant $c > 0$ and function $f : \mathbb{N} \to \mathbb{R}$ show that $O(cf(n)) = O(f(n))$.
It is sufficient to show (i) $cf(n) \in O(f(n))$ and (ii) $f(n) \in O(cf(n))$.

(b) Show that $O(\log_a n) = O(\log_b n)$. The easiest way to do this is to convert both logarithms to base $e$ using $\log_a n = \frac{\log_e n}{\log_e a} = \frac{\ln n}{\ln a}$.

(c) Simplify the following

(i) $2nO(2n + 1)$

(ii) $O(2(n + 1)^2 + 3n)$,

(iii) $O(n^2) + nO(n^2/3)$ and

(iv) $O(2^n + n^3)$.

No justification required.

## Question 4 : The Euclidean Algorithm [10 marks]

Given $a, b \in E$, a Euclidean domain, the extended Euclidean algorithm solves $sa + tb = g$ for $s, t \in E$ and $g = \gcd(a, b)$.

(a) For integers $a = 99$, $b = 28$ execute the extended Euclidean algorithm by hand. Use the tabular method presented in class that shows the values for $r_k, s_k, t_k, q_k$. Identify $b^{-1} \bmod a$.

(b) For integers $a = 1234$ and $b = 4321$ use Maple's `igcdex` command to find integers $s$ and $t$ such that $sa + tb = g$ where $g = 1$. Identify $a^{-1}$ modulo $b$. Check your answer by calculating $a^{-1} \bmod b$ in Maple.

(c) For polynomials $a = x^3 - 1$ and $b = x^4 - 1$ use Maple's `gcdex` command to find polynomials $g$, $s$ and $t$ in $\mathbb{Q}[x]$ such that $sa = tb = g$ where $g$ is the monic gcd of $a, b$.

## Question 5 : Polynomial Interpolation [27 marks]

Let $F$ be a field and let $x \in F^n$ and $y \in F^n$ be $n$ points. In class I presented Lagrange interpolation to interpolate the unique polynomial $f(x)$ of degree at most $n-1$ such that $f(x_i) = y_i$ for $1 \leq i \leq n$. It is based on the Lagrange basis. Let $L = \prod_{i=1}^{n}(x - x_i)$. The Lagrange basis is

$$\{L_i(x) = \frac{L(x)}{x - x_i} \text{ for } 1 \leq i \leq n.\}$$

Notice that each $L_i(x)$ has degree $n-1$. The interpolating polynomial $f(x)$ is given by

$$f(x) = \sum_{i=1}^{n} \alpha_i L_i(x)$$

where the constants $\alpha_i = y_i / L_i(\alpha_i)$.

(a) Prove that the Lagrange basis polynomials $L_i(x)$ are linearly independent in $F[x]$.

(b) By hand, using both Newton interpolation and Lagrange interpolation, find $f(x) \in \mathbb{Q}[x]$ such that $f(0) = 1, f(1) = 3, f(2) = 4$ such that $\deg(f) \leq 2$.

(c) Here is how we can compute $f(x)$ using Lagrange interpolation.

Step 1 Compute $L(x) = \prod_{i=1}^{n}(x - x_i)$ in expanded form.

Step 2 Compute the Lagrange basis polynomials $L_i(x) = L(x)/(x - x_i)$ for $1 \leq i \leq n$.

Step 3 Compute the Lagrange coefficients $\alpha_i = y_i / L_i(x_i)$.

Step 4 Compute and output the interpolating polynomial $f = \sum_{i=1}^{n} \alpha_i L_i(x)$.

Let $T(n)$ be the number of multiplications in $F$ that Steps 1 to 4 do. Find an exact formula for $T(n)$ and then express $T(n)$ in big O notation. Note, in step 1, each time you multiply by $x - x_i$, multiplication by $x$ does not need any multiplications in $F$. Note that step 2 is a polynomial division in $F[x]$ whereas step 3 is a scalar division in $F$. For step 2 work out how many multiplications polynomial long division by $x - x_i$ does. Step 4 is a scalar multiplication of $\alpha_i$ by a polynomial $L_i(x)$ of degree $n-1$.

(d) Write a Maple procedure that implements Lagrange interpolation for $F = \mathbb{Q}$. For the polynomial multiplications in step 1 use the `expand` command. For the polynomial divisions in step 2 use the `quo` command or the `divide` command. Test your procedure on the example from part (b) and on $x = [1, 2, 3, 4]$, $y = [-1, 2, 7, 14]$. Verify your answers by checking that $f(x_i) = y_i$ for $1 \leq i \leq n$.