A1Q4 Solution, Copyright Michael Monagan 2023

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(a) Prove that the Lagrange basis polynomials

$$L_i(x) = \prod_{\substack{j=1 \\ j \neq i}}^{n} x - \alpha_i \quad \text{where } \alpha_i \in F \text{ are distinct.}$$

are linearly independent in $F[x]$.

Notice that the condition α_i are distinct $\Rightarrow$ $L_i(\alpha_i) \neq 0$.

Consider

$$\underbrace{c_1 L_1(x) + c_2 L_2(x) + \cdots + c_n L_n(x)}_{S(x).} = 0 \quad \text{for some } c_i \in F.$$

We have

$$S(\alpha_1) = c_1 \cdot \overbrace{L_1(\alpha_1)}^{\neq 0} + c_2 \cdot 0 + \cdots + c_n \cdot 0 = 0 \Rightarrow c_1 = 0.$$

$$S(\alpha_i) = \sum_{j=1}^{i-1} c_j \cdot 0 + c_i L_i(\alpha_i) + \sum_{j=i+1}^{n} c_j \cdot 0 = 0$$

$$\Rightarrow \quad c_i L_i(\alpha_i) = 0 \Rightarrow c_i = 0 \text{ as } L_i(\alpha_i) \neq 0.$$

Thus $S(x) = 0 \Rightarrow c_i = 0 \Rightarrow \{ L_i(x) : 1 \leq i \leq n \}$ are Linearly independant by definition.

(b) We are given $n=3$ points
$\alpha_1, y_1 = 0, 1$
$\alpha_2, y_2 = 1, 3$
$\alpha_3, y_3 = 2, 4.$

<u>Newton's method:</u>

$$f = b_0 + b_1(x - \alpha_1) + b_2(x - \alpha_1)(x - \alpha_2)$$
$$f = b_0 + b_1(x) + b_2 x(x-1).$$

$$y_1 = f(\alpha_1) = 1 = b_0 + b_1 \cdot 0 + b_2 \cdot 0 \Rightarrow b_0 = 1$$
$$y_2 = f(\alpha_2) = 3 = 1 + b_1 \cdot 1 + b_2 \cdot 0 \Rightarrow b_1 = 2.$$
$$y_3 = f(\alpha_3) = 4 = 1 + 2 \cdot 2 + b_2 \cdot 2 \cdot 1 = 5 + 2b_1 \Rightarrow b_1 = -\tfrac{1}{2}$$

$$f(x) = 1 + 2x - \tfrac{1}{2}x(x-1)$$
$$= -\tfrac{1}{2}x^2 + \tfrac{5}{2}x + 1.$$

Lagrange's method

$$f = a_1(x-\alpha_2)(x-\alpha_3) + a_2(x-\alpha_1)(x-\alpha_3) + a_3(x-\alpha_1)(x-\alpha_2)$$

$$f = a_1\underbrace{(x-1)(x-2)}_{L_1(x)} + a_2\underbrace{x(x-2)}_{L_2(x)} + a_3\underbrace{x(x-1)}_{L_3(x)}.$$

$$y_1 = 1 = f(0) = a_1\cdot(-1)(-2) + 0 + 0 \Rightarrow a_1 = \tfrac{1}{2}$$
$$y_2 = 3 = f(1) = 0 + a_2\cdot 1\cdot(-1) + 0 \Rightarrow a_2 = -3$$
$$y_3 = 4 = f(2) = 0 + 0 + a_3 2\cdot 1 \Rightarrow a_3 = 2.$$

$$f = \tfrac{1}{2}(x-1)(x-2) - 3x(x-2) + 2x(x-1)$$
$$= \tfrac{1}{2}x^2 - \tfrac{3}{2}x + 1 - 3x^2 + 6x + 2x^2 - 2x$$
$$= -\tfrac{1}{2}x^2 + \tfrac{5}{2}x + 1.$$

(c) Step 1:

$L := 1$

for $i = 1, 2, \ldots, n$ do $L := \underset{\text{has degree } i-1}{L} \cdot (x-\alpha_i) = \underset{\text{no mults in F}}{x \cdot L} - \underset{i \text{ mults in F}}{\alpha_i \cdot L}$.

The # mults in F is $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

Step 2:

for $i = 1, 2, \ldots, n$ do $L_i = \frac{L(x) \leftarrow \text{degree } n}{x-\alpha_i \leftarrow \text{degree } 1}$

In the lecture I showed each division of L by $x-\alpha_i$ does at most n mults in F thus step 2 does $\leq n^2$ mults.

Step 3: does 0 mults in F.

Step 4: $f := \sum_{i=1}^{n} a_i \cdot L_i(x)$.

a_i: scalar mult. $L_i(x)$: degree $n-1$ so n coefficients.

This does n^2 mults in F.

Total $T(n) \leq \frac{1}{2}n^2 + \frac{1}{2}n + n^2 + n^2 = \frac{5}{2}n^2 + \frac{1}{2}n \in O(n^2)$ mults in F