MATH 895 Assignment 5, Fall 2021

Instructor: Michael Monagan

Please hand in the assignment by 11pm Saturday November 6th.
Late Penalty -20% off for up to 36 hours late. Zero after that.
For Maple problems, please submit a printout of a Maple worksheet containing Maple code and the execution of examples.

Question 1

(a) Let I and J be two ideals in $k[x_1, \ldots, x_n]$.
Prove or disprove that $I \cap J$ and $I \cup J$ are ideals.

(b) Let $I = \langle f_1, f_2, ..., f_s \rangle$ and $J = \langle g_1, g_2, ..., g_t \rangle$ be ideals in $k[x_1, \ldots, x_n]$.
Prove that $I = J \Longrightarrow \mathbb{V}(f_1, f_2, \ldots, f_s) = \mathbb{V}(g_1, g_2, ..., g_t)$.
Show that the converse is false in general.

(c) Let $I = \langle x - z, xy - 1, y - z \rangle$. Use the very useful lemma to simplify the basis for I then describe $\mathbb{V}(x - z, xy - 1, y - z)$.

Question 2

(a) In the definition of a monomial ordering, prove that (iii) $<$ is a well ordering is equivalent to (iii) the least monomial under $<$ is 1.

(b) Consider $f_1 = x + y^2 - 1$, $f_2 = xy - 1$ and $f = y^3 + 2xy - y - 1$. Use the division algorithm to divide f by $\{f_1, f_2\}$ using >lex and >grlex with $x > y$.

(c) State the Ascending Chain Condition. To complete the proof let $I_1, I_2, I_3, \ldots$ be ideals in $k[x_1, \ldots, x_n]$ s.t. $I_1 \subset I_2 \subset I_3 \subset \ldots$ Prove that $\bigcup_{i=1}^{\infty} I_i$ is an ideal in $k[x_1, \ldots, x_n]$.

Question 3

(a) Consider the ideals $\langle xy - z, xz - y \rangle$ and $\langle x + 1, xy + 1, y - 1 \rangle$.
Use Buchberger's S-polynomial criterion to determine which are Groebner bases.
For the Groebner bases, are they minimal? reduced? Assume >lex with $x > y > z$.

(b) Consider $I = \langle x - z^2, y - z^3 \rangle$. Compute a Groebner basis for I using Buchberger's algorithm wrt <lex with $x > y > z$ and >grlex with $x > y > z$. Make your basis a reduced Groebner basis. Do this by hand. Use Maple to check your calculations.

(c) Consider $I = \langle xy - 1, xz - 1, yz - 1 \rangle$. Using Maple's NormalForm and SPolynomial commmands, compute a Groebner basis G for I using >grlex order with $x > y > z$.
What is $\langle LT(I) \rangle$? If $f \in k[x, y, z]$, which monomials can appear in the remainder of $f \div G$?

Question 4

(a) Suppose we have numbers x, y, z that satisfy

$$x + y + z = 3, x^2 + y^2 + z^2 = 5, x^3 + y^3 + z^3 = 7.$$

Use Groebner bases to prove that $x^4 + y^4 + z^4 = 9$. Do this by testing if $x^4 + y^4 + z^4 - 9$ is in the ideal $\langle x + y + z - 3, x^2 + y^2 + z^2 - 5, x^3 + y^3 + z^3 - 7\rangle$. What is $x^5 + y^5 + z^5$? Use Maple for this question.

(b) Consider the function $f(x, y, z) = x^3 + 2xyz - z^2$. Use the method of lagrange multipliers to find the maximum of $f(x, y, y)$ subject to the constraint $x^2 + y^2 + z^2 = 1$. To do this form the Lagrangian function

$$L = f - \lambda g$$

and use an appropriate monomial ordering to solve $\{ L_x = 0, L_y = 0, L_z = 0, g = 0 \}$ for x, y, z. You should get 10 distinct solutions. One will be the maximum.

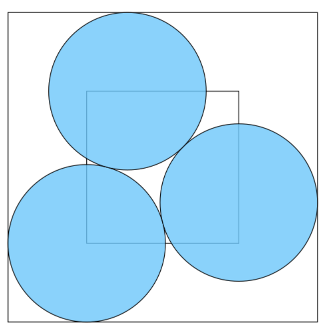
Question 5

(a) The trigonometric parametrization of a circle with radius r is given by

$$x(t) = r \cos t, y(t) = r \sin t \quad \text{for } 0 \leq t \leq 2\pi.$$

Consider the ideal $I = \langle x - rc, y - rs, s^2 + c^2 - 1\rangle$. Use Maple to compute a Grobner basis for the elimination ideal $I \cap \mathbb{Q}(r)[x, y]$ using an appropriate monomial ordering.

(b) In the figure below three circles of equal diameter m have been placed in the unit square with m maximal.



Let $P1 = (x_1, y_1)$, $P2 = (x_2, y_2)$ and $P3 = (x_3, y_3)$ be the centres of the three circles. Construct a system of 7 equations $\{f_1 = 0, f_2 = 0, ..., f_7 = 0\}$ in the seven unknowns $x_1, x_2, x_3, y_1, y_2, y_3, m$. Since the circles touch each other, you can apply Pythagoras' theorem to obtain three quadratic equations. You can get four more simple equations by noting where the circles touch the boundary of the unit square. Let $I = \langle f_1, f_2, ..., f_7\rangle$. Compute a Grobner basis for $I \cap \mathbb{Q}[m]$ using an appropriate monomial ordering. You should get a polynomial of degree 4 in m. Now solve this for m numerically in Maple and identify m.