SFU RESEARCH MATTERS

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Efficient Computational Methods for Water Waves via Boundary Integral Methods

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Introduction & Purpose

- Surface waves have interested a considerable number of mathematicians for many centuries and still intrigue us today.
- A lot has been done to study this nonlinear problem, yet not all is known.
- We need to understand the underlying mechanisms that influence them.
- Why? They are everywhere: sloping waves in the beaches, flood waves in rivers, free oscillations in lakes and harbors, fronts in the atmosphere, tsunami waves, to mention just a few.
- Computational Fluid Dynamicists have been recently investigating the myriad of challenging problems that come up from a new point of view.
- Why Water Waves? It is an interesting representative of the Free Boundary Problems and quite challenging on itself.

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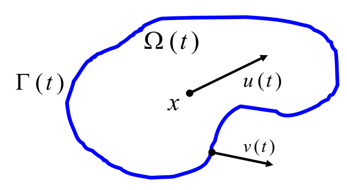
MITACS

Free Boundary Problems in Fluid Dynamics

They arise when the dynamics of:

- *The boundary of a fluid*
- *A boundary between two fluids*
- *A boundary within the fluid itself*

must be simultaneously determined with the dynamics of the fluid.



The boundary Γ changes in time, as well as the fluid domain Ω .

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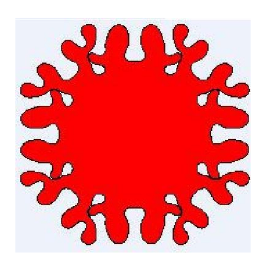
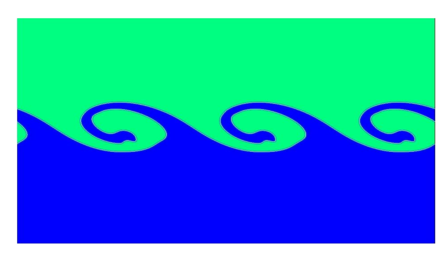
Examples of Free Boundary Problems

- **Inertial Flows**
  - Kelvin-Helmholtz instability with surface tension *(fig.1 – from H. Ceniceros' website)*
  - Rayleigh-Taylor instability
  - Capillary Waves *(will see this one later)*
  - Axisymmetric Flow
- **Stokes' Flow** – Bubbles and drops in viscous flow.
- **Hele - Shaw Flows**
  - Pattern Formation/ Selection *(fig. 2)*
  - Surface Tension driven Singularity Formation
  - Taylor-Saffman Instability *(fig. 2 – from M. Shelley's website)*
  - Axisymmetric Porous Flow
- **Materials' Science**





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Why are these problems interesting and difficult?

- Flows in the domain Ω are non-local, e.g. the continuity equation $\nabla \bullet \vec{u} = 0$ is a global constraint on the flow.
- The flow domain Ω(t) is time-dependent, and so is the boundary Γ(t) and can become very ramified and even singular.
- Surface stresses can be complicated by nonlinear dependencies on the geometry (e.g. the surface tension is dependent on the curvature.

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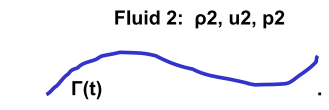
Equations and Boundary Conditions

For this derivation, consider the motion of a general 2-fluid flow in 2-D which is:

- ***Incompressible***
- ***Irrotational***
- ***Inviscid***
- ***Has Infinite Depth.***



Navier-Stokes Equations:

$$\frac{\partial u_i}{\partial t} + (u_i \bullet \nabla) u_i = -\frac{1}{\rho_i}\nabla p_i - g\vec{j}$$

$$\nabla \bullet u_i = 0$$

Boundary Conditions:

$[u]_\Gamma \bullet \vec{n} = 0$ **Kinematic Condition**

$[p]_\Gamma = \tau\kappa$ **Laplace-Young Condition**

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The Variables & Equations in the Boundary Integral Formulation

Interface position: z(α,t) = x(α,t)+iy(α,t)
Complex Velocity: W(α,t) = u(α,t)-ivα,t)
Complex Potential: Φ(α,t) = Φ(α,t) +iΨ(α,t)
Vortex Sheet Strength: γ(α,t)
Curvature: κ(α,t)

$$\frac{dz^*}{dt} = W = u - iv = \frac{d\Phi}{dz} = \frac{1}{4\pi i}\int_0^{2\pi} \gamma(\alpha')\cot(\frac{z(\alpha)-z(\alpha')}{2})d\alpha'$$

$$\phi_t - \frac{1}{2}|W|^2 + gy = \tau\kappa \qquad \kappa = \frac{y_{\alpha\alpha}x_\alpha - x_{\alpha\alpha}y_\alpha}{(x_\alpha^2 + y_\alpha^2)^{3/2}}$$

$$\phi_\alpha(z) = \frac{\gamma(\alpha)}{2} + \mathrm{Re}\frac{z_\alpha(\alpha)}{4\pi i}\int_0^{2\pi}\gamma(\alpha')\cot(\frac{z(\alpha)-z(\alpha')}{2})d\alpha'$$

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Numerics & Implementation

- Numerical simulations using Boundary Integral Methods are sensitive to numerical instabilities.
- A compatibility between the choice of quadrature rule for the singular integral and the discrete derivatives must be satisfied.
- For Spectral Accuracy we choose:
  - Pseudospectral Approximations for the Space Derivatives
  - Alternating Trapezoidal Rule for the Singular Integral
- Integrator for the ODE-s: 4th order Adams-Bashforth or Runge-Kutta method.
- GMRES algorithm is used to solve iteratively for γ.
- A 4th order extrapolation method in time is used to obtain a more accurate initial guess for γ before solving iteratively for it.
- Doubling of points is often needed when the wave enters the breaking regime.

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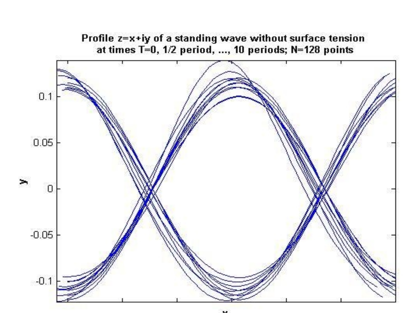
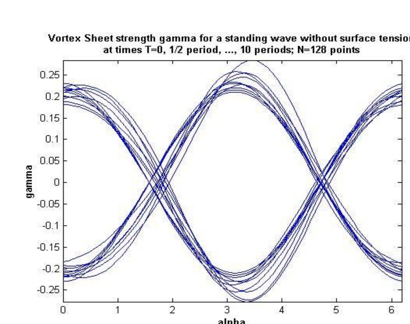
Standing wave without Surface Tension

To check that algorithm works correctly, a standing wave is computed. We expect the interface position not to differ much after each period (2π) in the limit the amplitude of the wave is much smaller than the depth.



profile z of the wave



vortex sheet strength γ

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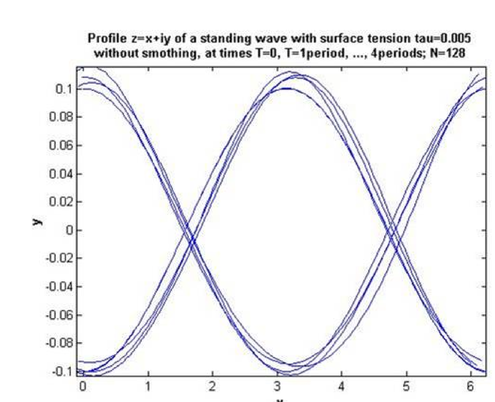
Standing wave with Surface Tension



When surface tension is added in the equation, the computations require a smaller time step, but with filtering, the wave can be advanced for quite a few periods.

We notice that the period of the standing wave is slightly shorter than when without surface tension.

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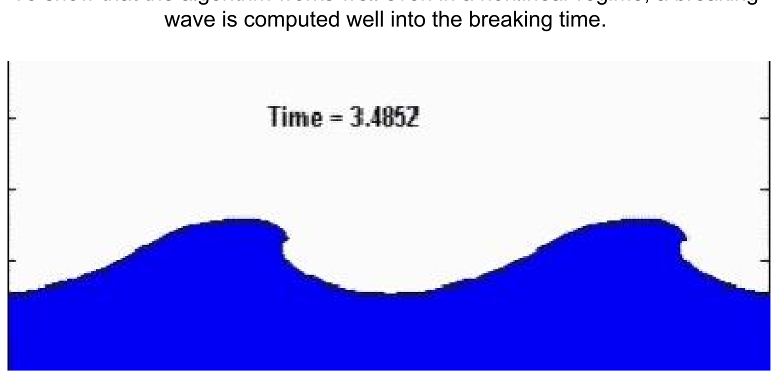
Breaking wave without Surface Tension

To show that the algorithm works well even in a nonlinear regime, a breaking wave is computed well into the breaking time.



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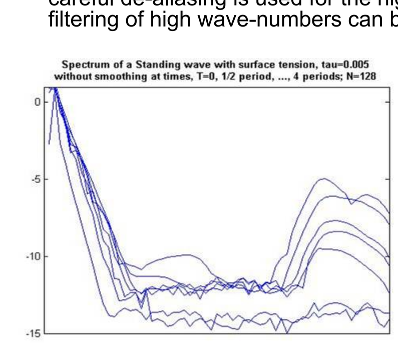
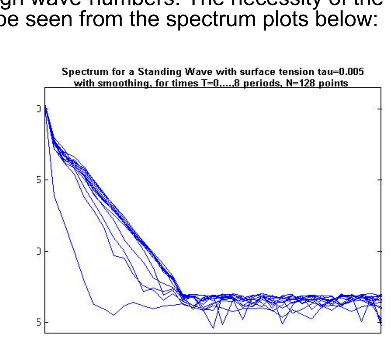
De-aliasing Issues

Beale, Hou, Lowengrub proved in 1996 the method is convergent provided careful de-aliasing is used for the high wave-numbers. The necessity of the filtering of high wave-numbers can be seen from the spectrum plots below:



Computations without filtering



Computations with filtering

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Problems that Arise

- When surface tension is added, the computations are stiff.
- Solving iteratively for γ takes time as the number of iterations keeps increasing.
- Stability Constraint is of the form:

$$\Delta t < \frac{C}{\tau}(\tilde{\sigma}\Delta x)^{3/2} \qquad \tilde{\sigma} = \min_\alpha \sigma$$

This restriction is quite severe for the time-step and, because point clustering happens in a breaking wave and σ can get very small.

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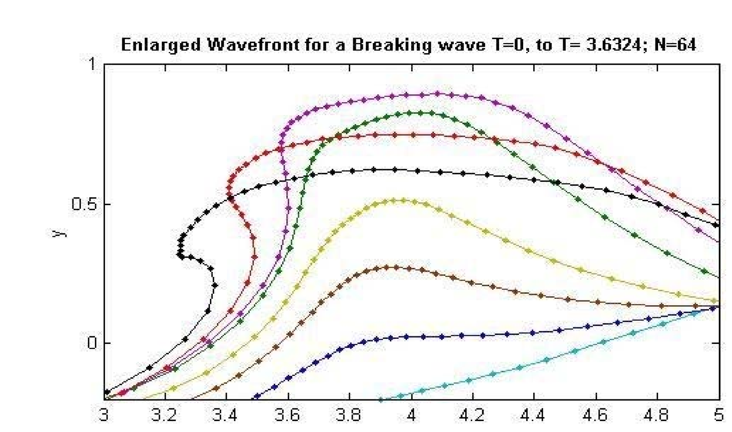
Point Clustering



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The fix: computations in the (σ,θ) frame

$$\frac{d\sigma}{dt} = -\overline{\theta_\alpha U^N}$$

$$\frac{d\theta}{dt} = \frac{1}{\sigma}U_\alpha^N + \frac{1}{\sigma}\theta_\alpha(U^T + U^A)$$

$$\frac{d\phi}{dt} = \frac{\tau}{\sigma}\theta_\alpha + \frac{1}{2}|W|^2 + U^A U^T - g\,\mathrm{Im}(z)$$

Where the velocity components are:

$$U^N = -|z_\alpha|^{-1}\,\mathrm{Im}(z_\alpha W) \qquad U^T = |z_\alpha|^{-1}\,\mathrm{Re}(z_\alpha W)$$

$$U^A = -U^A + \int_0^\alpha [\theta_\alpha U^N - \overline{\theta_\alpha U^N}]d\alpha'$$

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Small Scale Decomposition (S.S.D.)

The dominant terms are the curvature κ in the Bernoulli Equation and δUn/δα in the evolution equation for θ.

It can be shown that:

$$U^N = \frac{1}{2\sigma}H(\gamma) + R(\gamma) \qquad \gamma \approx 2\phi_\alpha$$

Then the evolution equations are written as:

$$\frac{d\theta}{dt} = \frac{1}{\sigma^2}H(\phi_{\alpha\alpha}) + P$$

$$\frac{d\phi}{dt} = \frac{\tau}{\sigma}\theta_\alpha + Q \qquad \frac{d\sigma}{dt} = -\overline{\theta_\alpha U^N}$$

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Implementation Issues

- Need to have an initial wave profile with equally spaced points to start with.
- The equations for θ and φ can be integrated using IMEX methods, since they decouple nicely in Fourier space and are easy to implement.
- Still need γ for the velocity W, and we solve for it iteratively, again. Extrapolation in time helps.
- GMRES is used for efficiently computing the integral equations for γ.
- Fast Multipole Methods should be used to evaluate the velocity, but this has not yet been incorporated into the algorithm.

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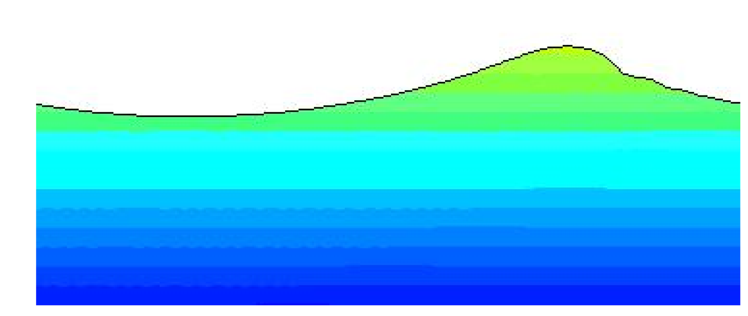
Capillary Waves' Appearance

When surface tension is added in the computations of the breaking wave, we notice capillary waves start to appear at the tip of the breaker.



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Benefits of Small Scale Decomposition

- Computations are not stiff, the time stepping constraint is not as severe as before.
- The points are equi-spaced, hence no point clustering occurs, if Ua is computed accordingly.
- Less computational time is needed.
- The interface is well-resolved, even in the breaking regime.

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Summary and Conclusions

- The understanding of the movement of water waves and their underlying mechanisms is important to the Mathematics/Engineering Community, but might have an impact in the Industry.
- The problem is difficult to derive and challenging to implement numerically.
- There are quite a few numerical stability issues, but computations in the equal arc-length frame help levitate some.
- The effects of the surface tension on the waves can be noticed.
- Capillary waves, resulting from gravity and surface tension effects, can be seen in the equal-arc-length computations of the breaking waves.

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Future Work

- Get a fully working code for the Small Scale Decomposition computations.
- Use a higher order semi-implicit scheme.
- Computations over a finite-depth topography with different profiles, with and without surface tension, to simulate shallow water waves.
- Incorporate weak viscosity in the equations and see what happens. We expect it to dampen the effect of the capillary waves.
- Do you have any suggestions?

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On the funny side, my computations have a long way to go before they catch up with things like this:

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